

# Stocks & Flows 2: Structure & Behavior

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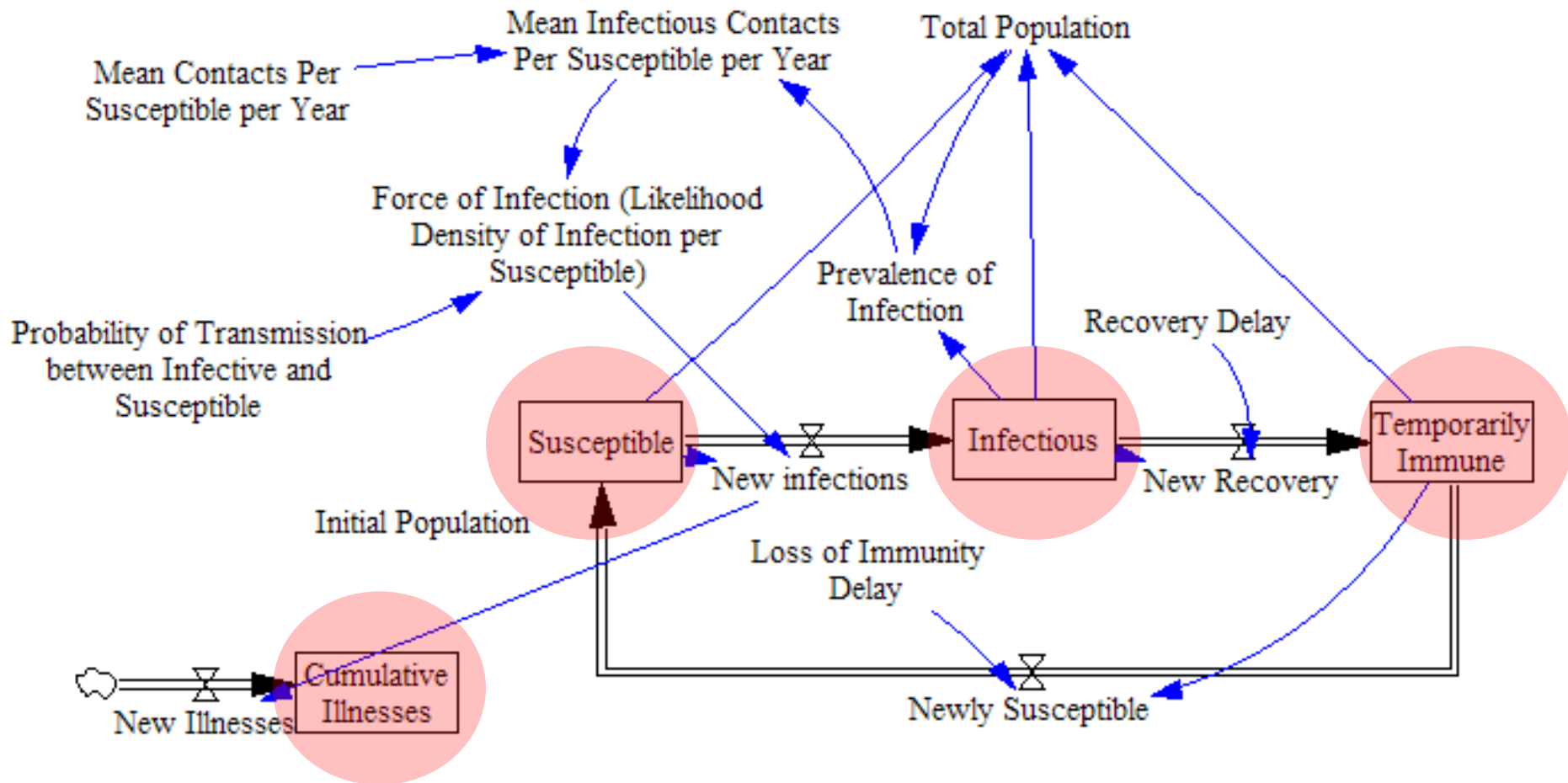
CMPT 394

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## Recall: State of the System: Stocks (“Levels”, “State Variables”, “Compartments”)

- Stocks (Levels) represent accumulations
  - These capture the “state of the system”
  - Mathematically, we will call these “state variables”
- These can be measured at *one instant in time*
- Stocks start with some initial value & are thereafter changed only by *flows* into & out of them
  - There are no inputs that immediately change stocks
- Stocks are the source of delay in a system
- In a stock & flow diagram, shown as ***rectangles***

# Example Model: Stocks



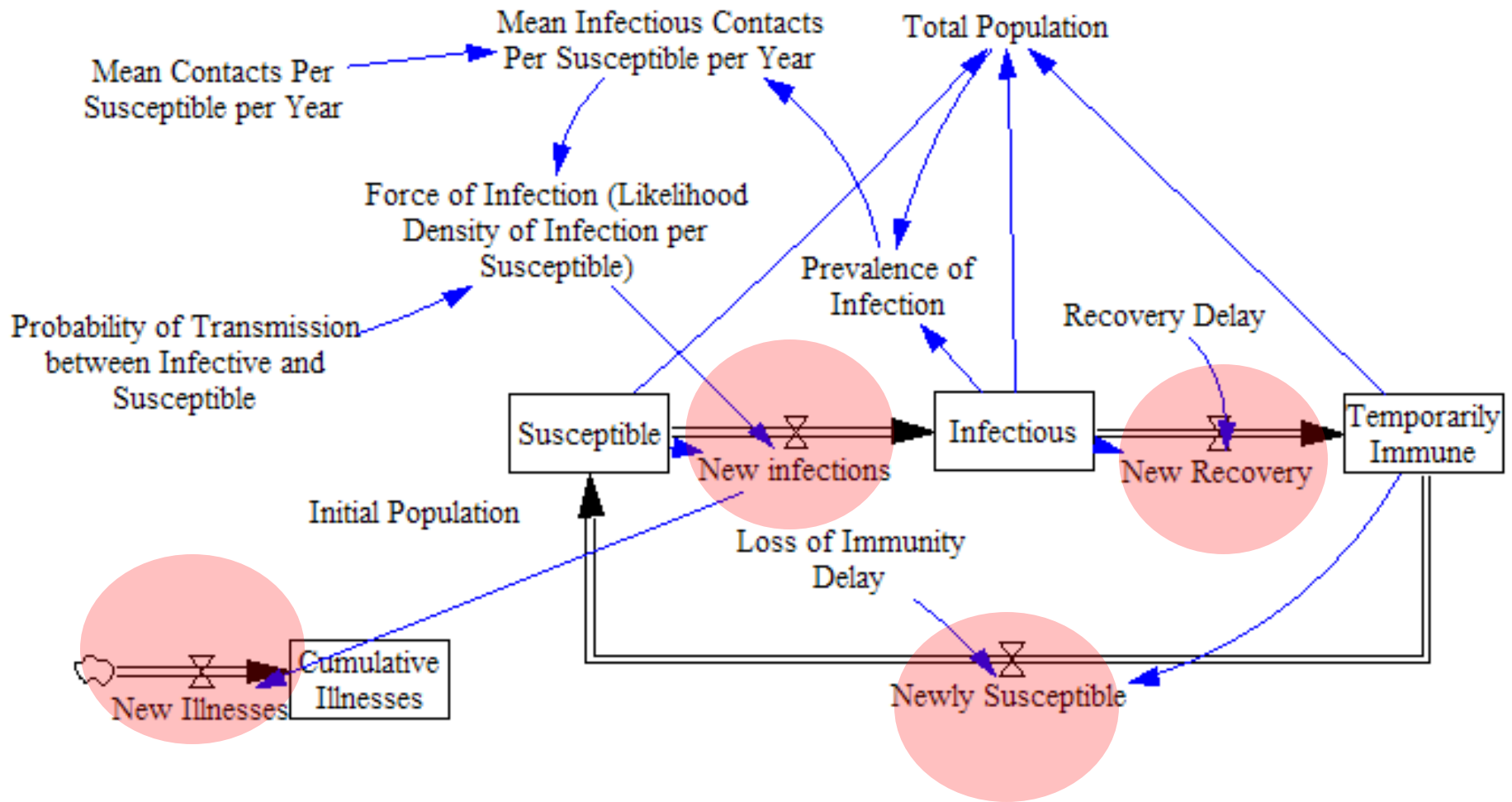
# Recall: The Critical Role of Stocks in Dynamics

- Stocks determine current state of system
  - Stocks often provide the basis for making choices
- Stocks central to most disequilibria phenomena (buildup, decay)
- Lead to inertia
- Give rise to delays

# Recall: State Changes: Flows (“Fluxes”, “Rates”, “Derivatives”)

- All changes to stocks occur via *flows*
- Always expressed per some unit time: If these flow into/out of a stock that keeps track of things of type  $X$  (e.g. persons), the rates are measured in  $X/(\text{Time Unit})$  (e.g. persons/year, \$/month, gallons/second)
- Typically measure over certain period of time (by considering accumulated quantity over a period of time)
  - e.g. Incidence Rates is calculated by accumulating people over a year, revenue is \$/Time, water flow is litres/minute
  - Can be estimated for any point in time

# Example Model: Flows

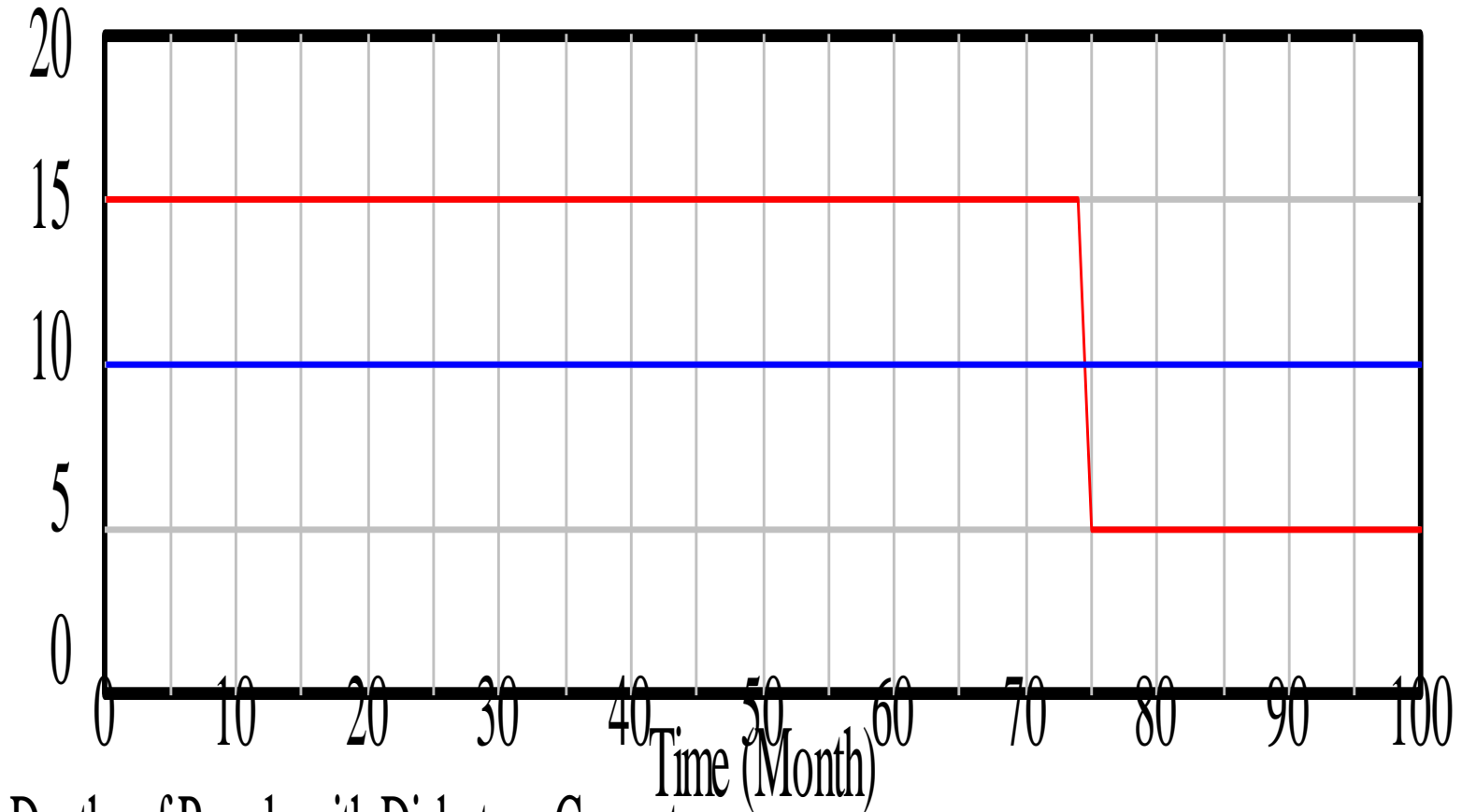


# Structure as Shaping Behaviour

- System structure is defined by
  - Stocks
  - Flows
  - Connections between them
- Nonlinearity: The behaviour of the whole is more than the sum of the behaviour of the parts
  - “Emergent” behaviour would not be anticipated from simple behaviour of each piece in turn
- Stock and flow structure (including feedbacks) of a system determines the qualitative behaviour modes that the system can take on

# Constant Flows

Diabetes Flows



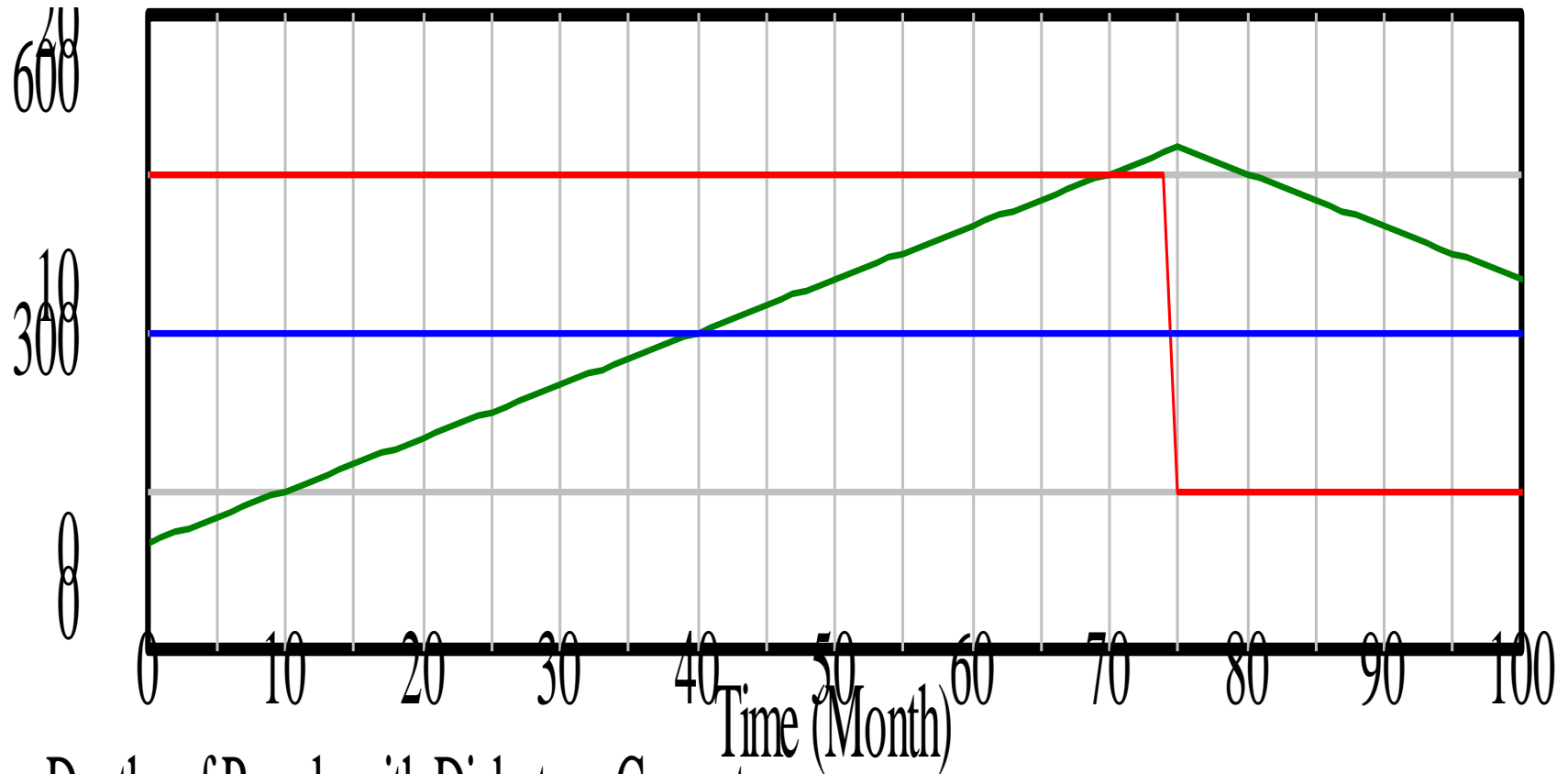
Deaths of People with Diabetes : Current

Incident cases of Diabetes : Current

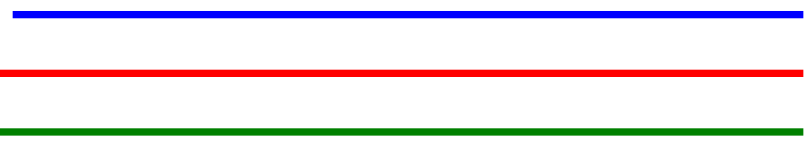
**What happens to the stock?**



# Resulting Stock (Green)

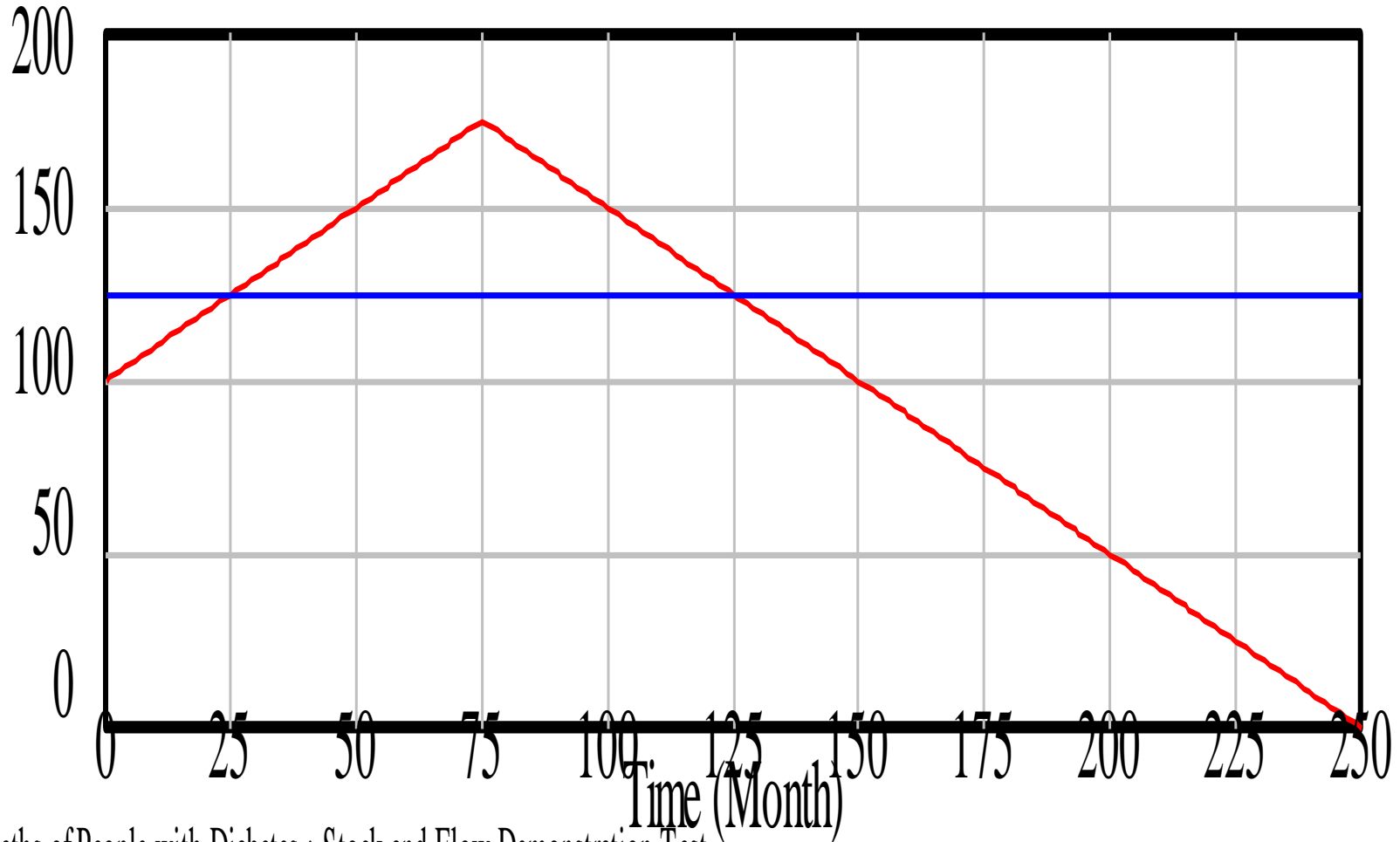


Deaths of People with Diabetes : Current  
Incident cases of Diabetes : Current  
People with Diabetes : Current



# Suppose we have these Flows (Rates)

## Diabetes Flows

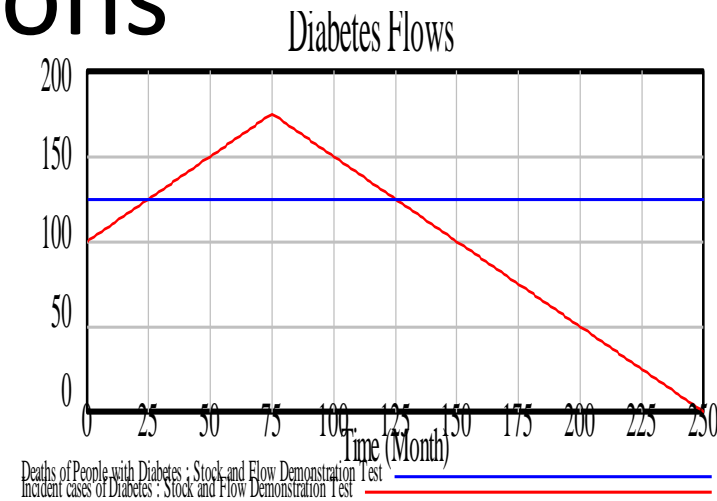


Deaths of People with Diabetes : Stock and Flow Demonstration Test  
Incident cases of Diabetes : Stock and Flow Demonstration Test

**What happens to the stock?**

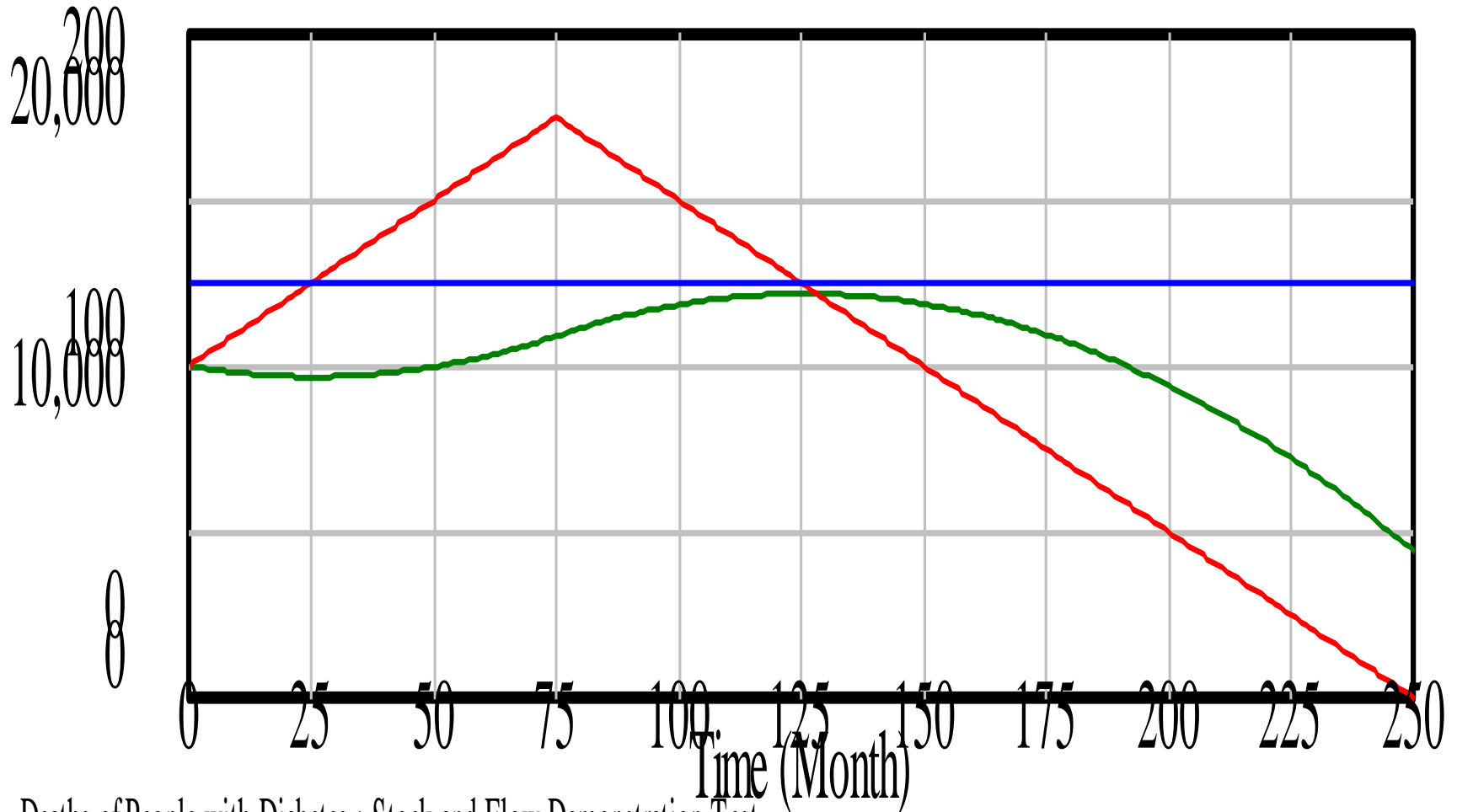
# Some Questions

- When is the stock of people with diabetes at its lowest value?
- When is the stock of people with diabetes at its greatest value?
- Is the value of the stock of people with diabetes larger at the beginning or end?
- When is the stock of people with diabetes not changing?



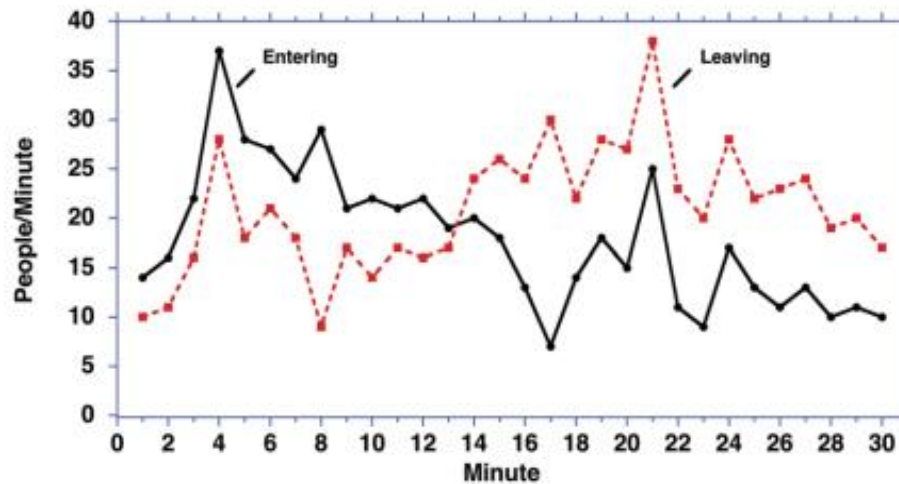
# Stock (Green)

## Diabetes Stock & Flows



Deaths of People with Diabetes : Stock and Flow Demonstration Test  
Incident cases of Diabetes : Stock and Flow Demonstration Test  
People with Diabetes : Stock and Flow Demonstration Test

The graph below shows the number of people *entering* and *leaving* a department store over a 30-minute period.



Please answer the following questions.

Check the box if the answer cannot be determined from the information provided.

1. During which minute did the most people enter the store?

Minute \_\_\_\_\_

Can't be determined

2. During which minute did the most people leave the store?

Minute \_\_\_\_\_

Can't be determined

3. During which minute were the most people in the store?

Minute \_\_\_\_\_

Can't be determined

4. During which minute were the fewest people in the store?

Minute \_\_\_\_\_

Can't be determined

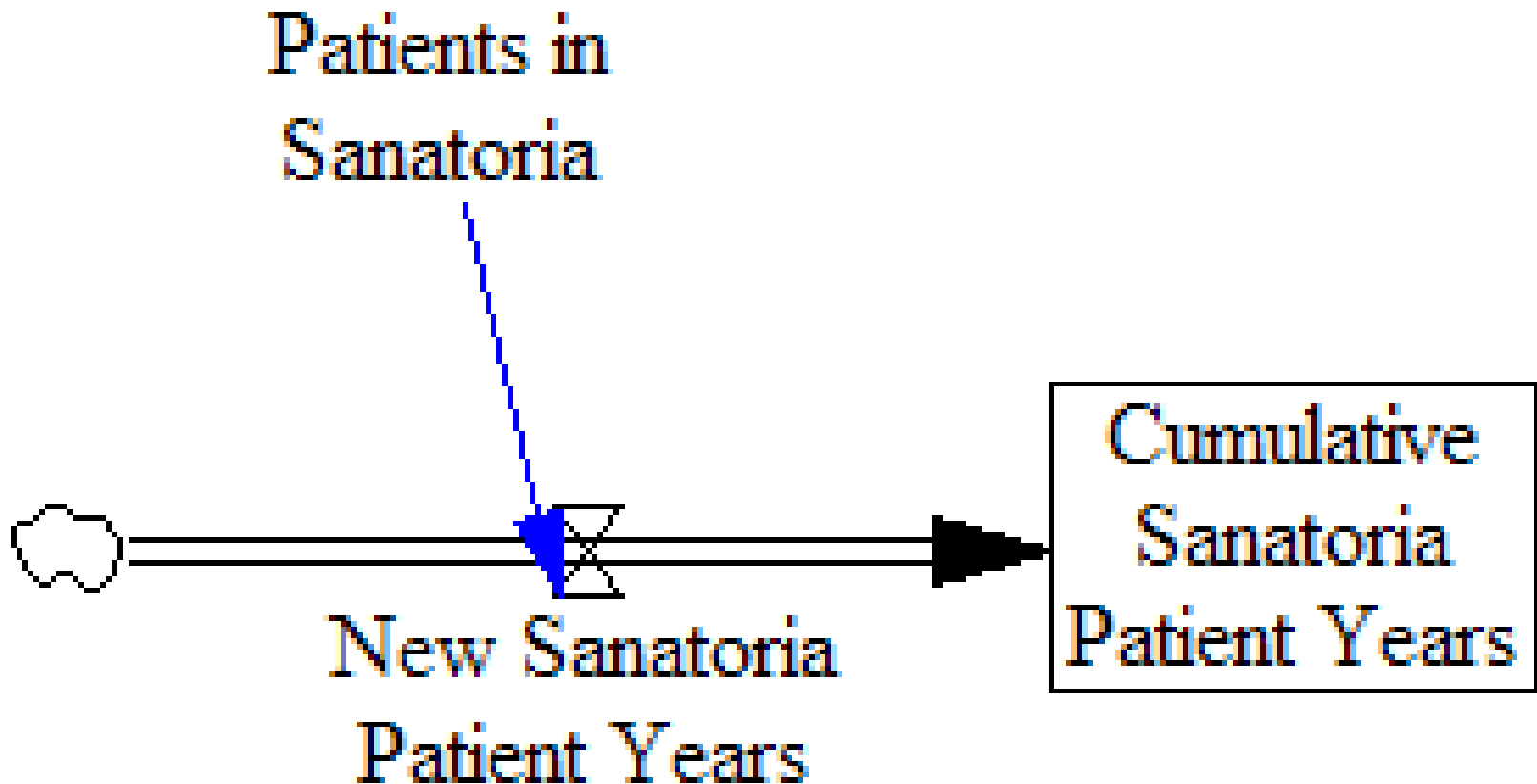
# Answers Among MIT Grad Students

- Questions 1&2: Vast majority correct (96% & 95%, respectively)
- Question 3: 44% correct
- Question 4: 31% correct

# Flows and Feedbacks

- Stocks are always changed by flows
- In your experiments, we've used constant values for flows
- In general, the formulas for the flows will depend on things that are changing (state)
  - Ultimately, these things must depend on the things that collectively specify the state – the stocks!

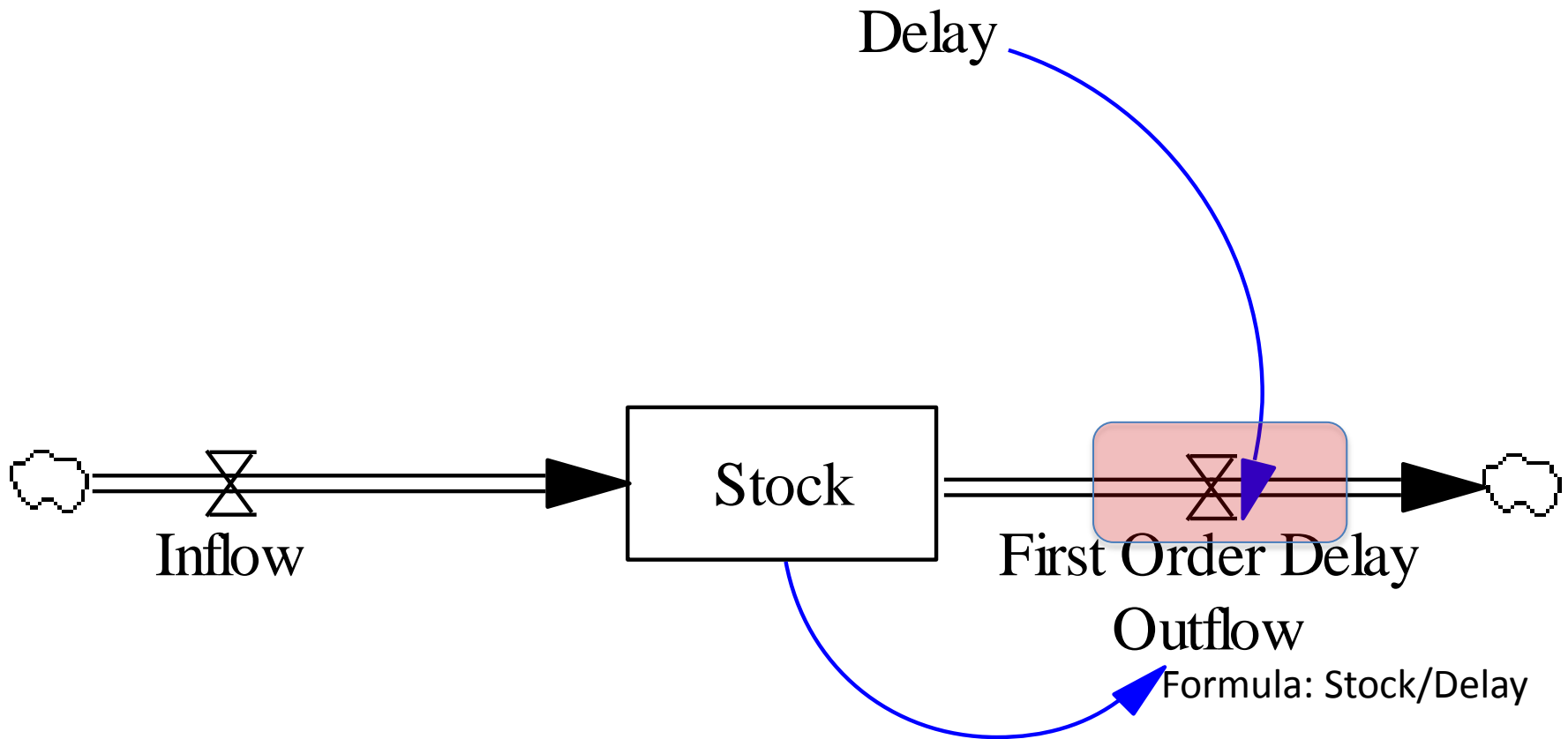
# Example 2





# Diabetes Model Stocks & Flows

(For a Challenge, Try Creating this in Vensim!)



# Equilibrium Value of a First-Order Delay

- Suppose we have flow of rate  $i$  into a stock with a first-order delay out
  - This could be from just a single flow, or many flows
- The value of the stock will approach an equilibrium where inflow=outflow

# Equilibrium Value of 1<sup>st</sup> Order Delay

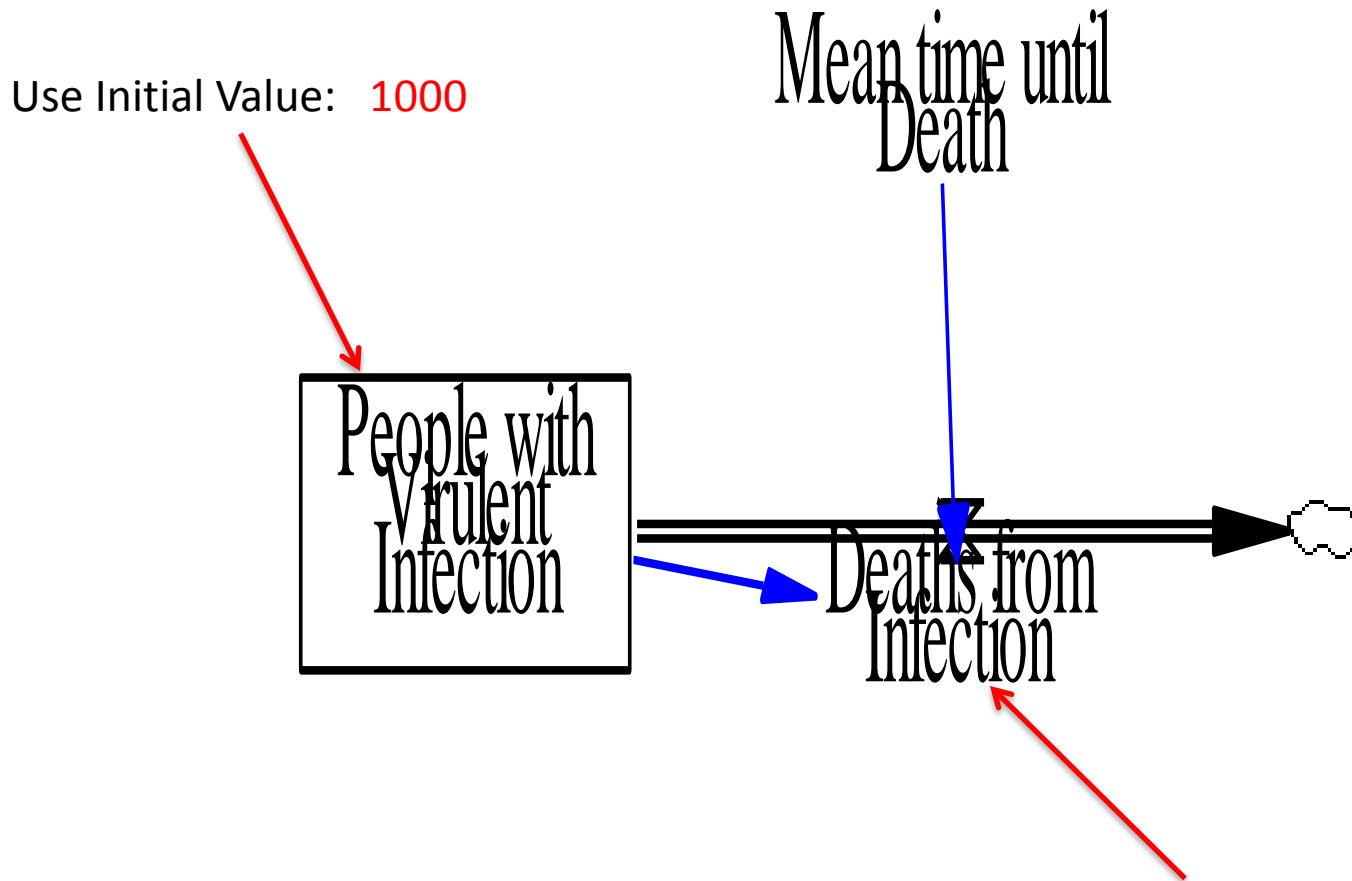
- Recall: Outflow rate for 1<sup>st</sup> order delay =  $\alpha x$ 
  - Note that this depends on the value of the stock!
- *Inflow rate =  $i$*
- At equilibrium, the level of the stock must be such that inflow = outflow
  - For our case, we have

$$\alpha x = i$$

Thus  $x = i/\alpha$

The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow = inflow

# Simple First-Order Decay (Create this in Vensim!)

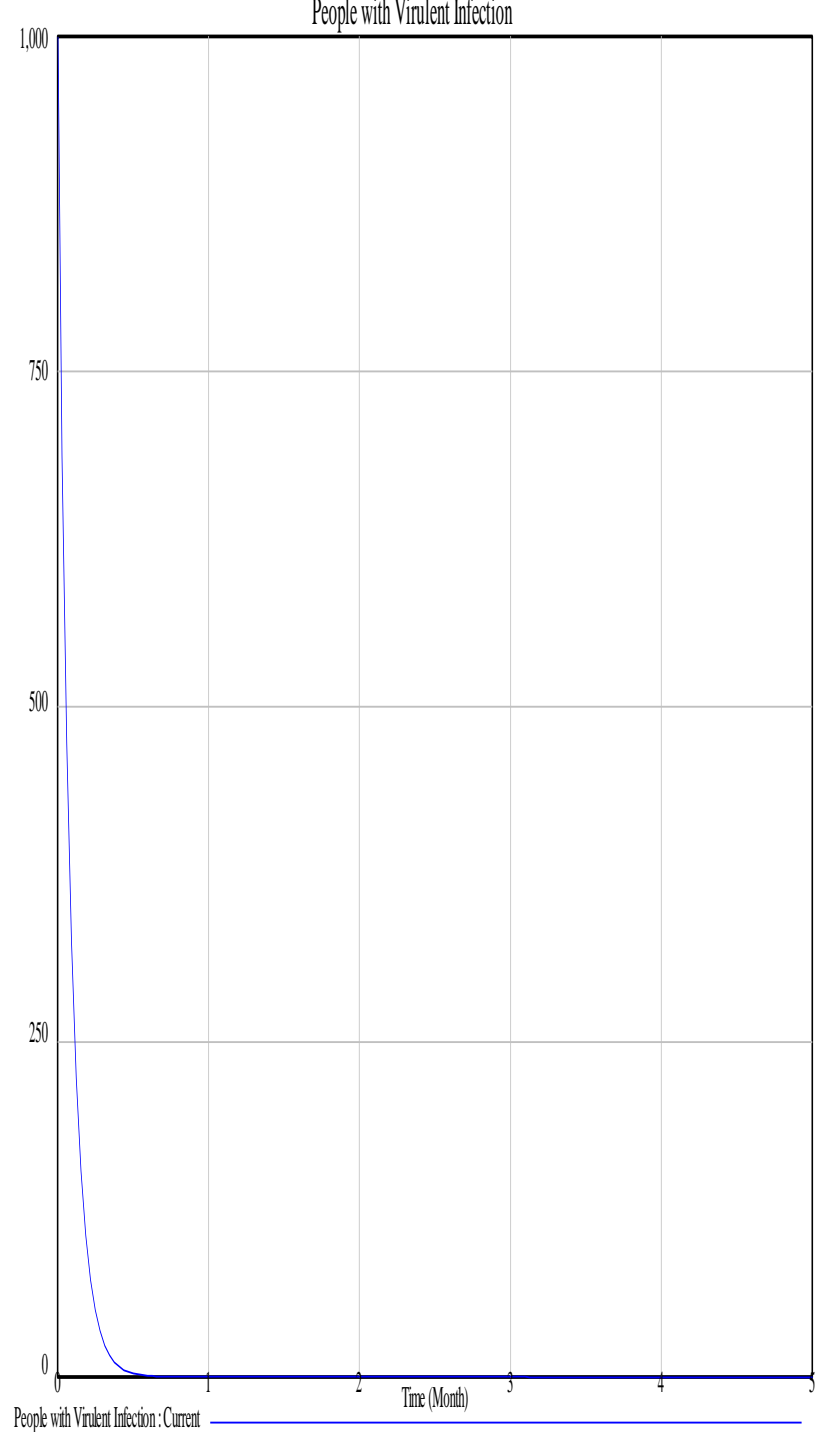


Use Formula:  $\text{People with Virulent Infection} / \text{Mean time until Death}$

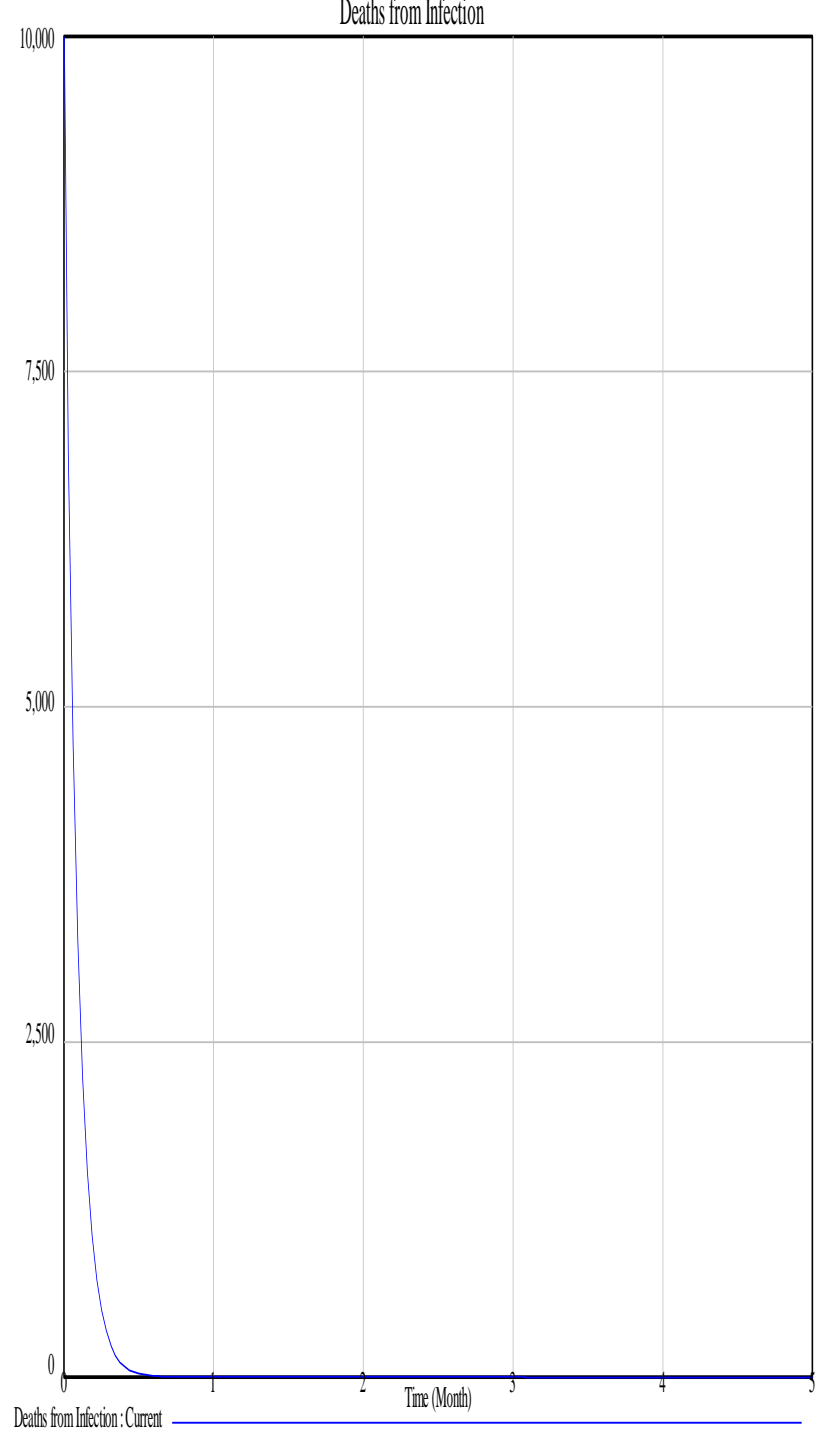
# First Order Delays and Transition Processes

- We can think of first order delays as representing a deterministic approximation to a population experiencing a memoryless (Poisson) stochastic transition process
- The system is “memoryless” because the chance of e.g. a person leaving in the next unit of time is independent of how long they’ve been there!
- The probability distribution of residence time in the stock is exponentially distributed

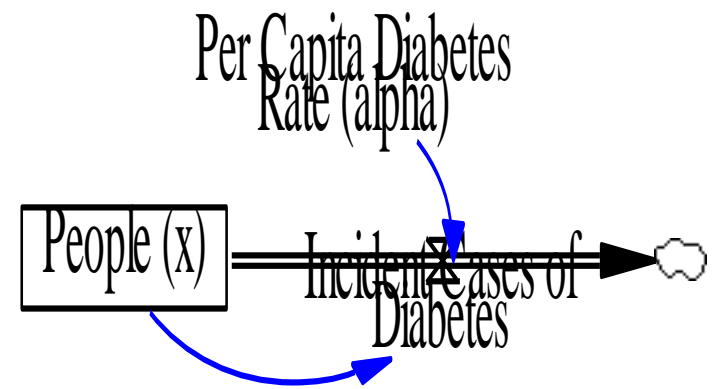
# Dynamics of Stock?



# Dynamics of (Rate of) Death Flow?



# Fundamental Continuous Mathematics (Calculus)



- Alpha is per-time-unit likelihood of death
    - Chance of death over small  $dt$  is  $\alpha dt$
    - If  $x$  people are at risk, # dying over  $dt$  is  $x * (\text{Likelihood of death over } \Delta t) = x(\alpha dt) = x\alpha dt$
    - When people die, they flow out  $\Rightarrow$  cause a negative change in  $x$ .
    - We denote the change in  $x$  over the time  $dt$  as  $\Delta x$
- Thus  $dx = -x\alpha dt$
- As  $x$  is depleted (becomes smaller),  $dx$  becomes smaller as well (for a fixed  $dt$ )



# Flow Rate Dynamics: Continuous

- The total change in  $x$  over the time  $dt$  is  $dx$

Thus  $dx = -x\alpha dt$

– This might be 20 people over a timeframe of 0.1 year (~36.5 days)

- The *rate of change* of  $x$  over given time  $dt$  is  $dx/dt$

This is just the sum of all of the flows!

For system,  $dx/dt = (-x\alpha dt)/dt = -\alpha x = -\text{People} * \text{DeathRate}$

Because  $x$  (People) changes, this flow rate changes over the course of the time we are observing

- We will often write  $dx/dt$  as  $\dot{x}$ , thus:  $\frac{dx}{dt} = \dot{x} = -\alpha x$

# Flow Rate Dynamics

- The total change in  $x$  over the time  $\Delta t$  is  $\Delta x$

Thus  $\Delta x = -x\alpha\Delta t$

– This might be 20 people over a timeframe of 0.1 year (~36.5 days)

- The *rate of change* of  $x$  over given time  $\Delta t$  is  $\Delta x/\Delta t$

This is just the sum of all of the flows

For system,  $\Delta x/\Delta t = (-x\alpha\Delta t)/\Delta t = -x\alpha = -\text{People} * \text{DeathRate}$

Because  $x$  (People) changes, this flow rate changes over the course of the time we are observing

Suppose time is measured in years; then for our example above,  $\Delta x/\Delta t = 20/0.1 = 200$  people per year

# Approximate Dynamics (Euler Integration)

Suppose

$$x(0)=1000$$

$$\Delta t=1$$

$$\alpha=0.2$$

Time (t)	Stock Value (x)	Change in stock ( $\Delta x$ ) $-x * \text{Alpha} * \text{DeltaT}$
0	1000	-200
1	800	-160
2	640	-128
3	512	-102.4
4	409.6	-81.92
5	327.68	-65.536

# Approximate Dynamics: Net Flow Rate

Reminder: Suppose

Initial  $x=1000$

$\Delta t=1$

$\alpha=0.2$

Time (t)	Stock Value (x)	Change in stock ( $\Delta x$ ) $-x * \text{Alpha} * \text{DeltaT}$	Net Flow Rate= $\Delta x / \Delta t$ Here, $\Delta t=1$ , so $\Delta x / \Delta t = \Delta x / 1 = \Delta x$
0	1000	-200	-200
1	800	-160	-160
2	640	-128	-128
3	512	-102.4	-102.4
4	409.6	-81.92	-81.92
5	327.68	-65.536	-65.536

# Why is This Approximate?

- Our previous graphs used a value of  $\Delta t=1$
- In calculating the change ( $\Delta x$ ) from  $t$  to  $t+\Delta t$  (here,  $t+1$ ), we are assuming that the flow rate (people/year) *stays constant in that time*
  - Recall: In general, this flow rate will be determined by the value of stocks
  - So in assuming that the flow rate remains constant, we were basically assuming that the values of the stocks stay constant over time  $\Delta t$ 
    - For our system, given that the value of the stock  $x$  (People) declines by around 20% per time unit, this is not a very good assumption!

# How Can We Reduce the Error?

## Try a Smaller $\Delta t$

- Let's work forward for  $\frac{1}{2}$  of a year at a time instead of for a full year

$$x(0)=1000$$

$$\Delta t=.5$$

$$\alpha=.1$$

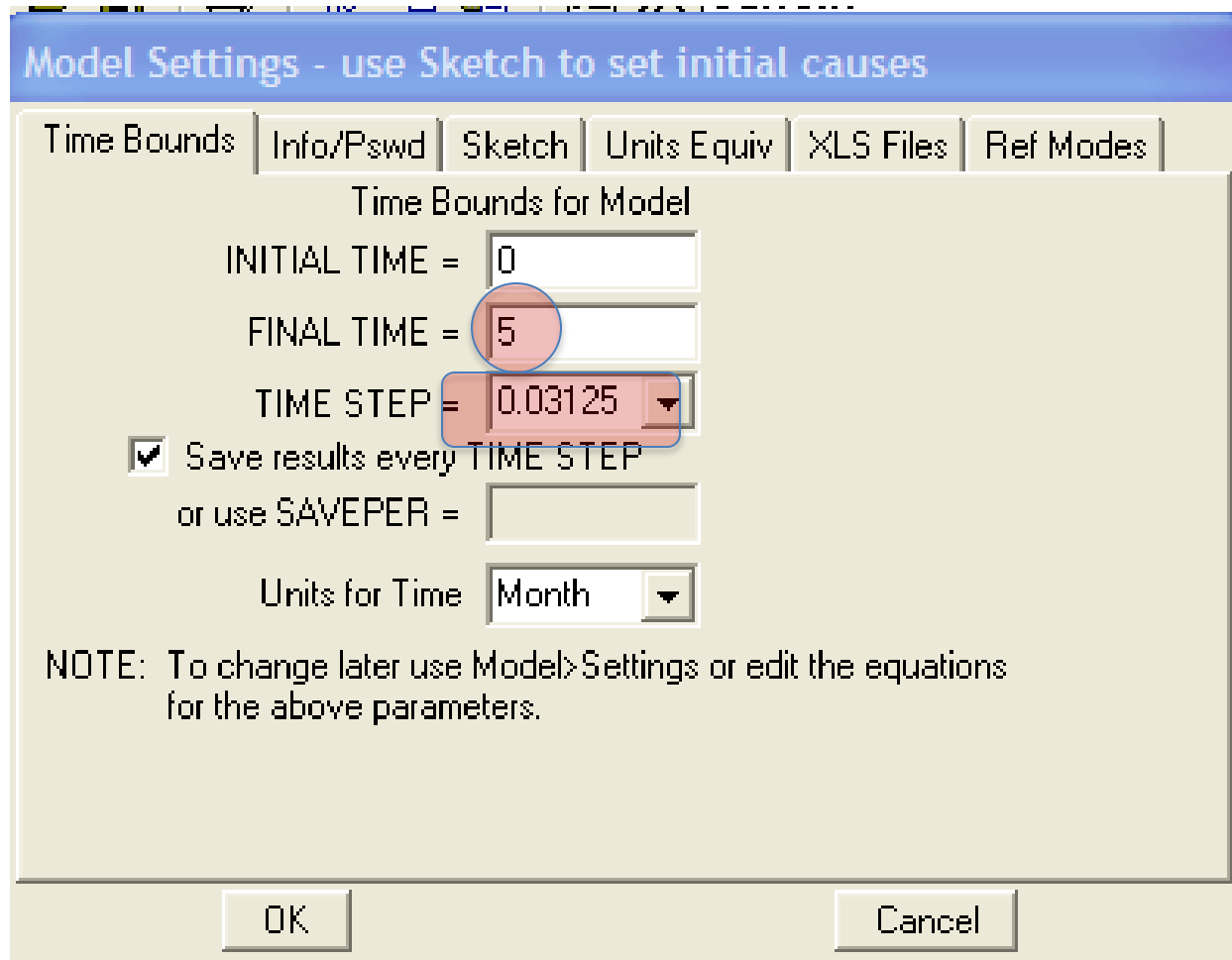
Time (t)	Stock Value (x)	Change in stock ( $\Delta x$ ) $-x*\text{Alpha}*\text{DeltaT}$	Net Flow Rate= $\Delta x/\Delta t$ Here, $\Delta t=1$ , so $\Delta x/\Delta t=\Delta x/1=\Delta x$
0	1000	-100	-200
0.5	900	-90	-180
1	810	-81	-162
1.5	729	-72.9	-145.8
2	656.1	-65.6	-131.2
2.5	590.5	-59.0	-118.1
3	531.4	-53.1	-106.3
3.5	478.3	-47.8	-95.7
4	430.5	-43.0	-86.1
4.5	387.4	-38.7	-77.5
5	348.7	-34.9	-69.7

# Approximate Dynamics: Net Flow Rate

	$\Delta t=1$	$\Delta t=.5$	$\Delta t=.25$
Time (t)	Stock Value (x)	Stock Value (x)	Stock Value (x)
0	1000	1000	1000
1	800	810	814.5
2	640	656.1	663.4
3	512	531.4	540.4
4	409.6	430.5	440.1
5	327.68	348.7	358.5

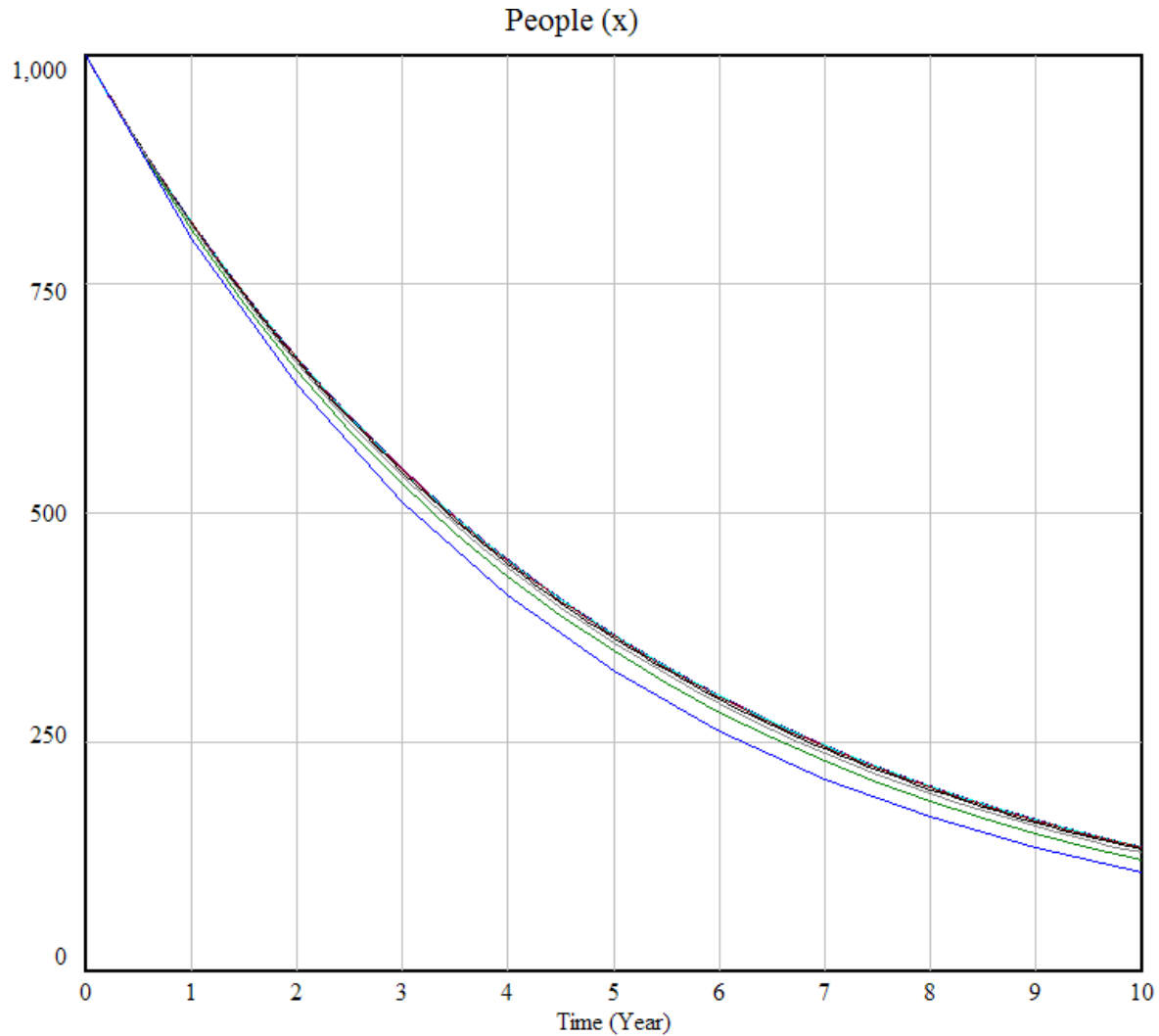
# Vensim & AnyLogic have a Step Size

## Vensim: (Set via Model Menu/Settings Item)





# Impact of Step Size on Simulation



"People (x)" : First order decay Timestep = 1  
"People (x)" : First order decay Timestep = pt1  
"People (x)" : First order decay Timestep = pt5  
"People (x)" : First order decay Timestep = pt25  
"People (x)" : First order decay Timestep = pt125  
"People (x)" : First order decay Timestep = pt0625  
"People (x)" : First order decay Timestep = pt03125  
"People (x)" : First order decay Timestep = pt015625  
"People (x)" : First order decay Timestep = pt007812

# The Concept of “Analytic” Solutions

- The model structure describes system behaviour *implicitly*
  - *This indicates how short term changes (flows) depends on the state of the system*
  - *This does not explicitly state how the system evolves*
- Analytic (“closed form”, “exact”) solutions describe system behaviour as an *explicit function of time*
  - *E.g.  $a+b*t+c*t^2$ ,  $a +b*t$ ,  $a*\sin(t)$ ,  $e^{\alpha t}$*
- For many systems we will be dealing with (nonlinear systems), an analytic solution *is simply not derivable*
  - Even when an analytic solution is possible, it is often most convenient to deal with simulations for most needs

# An Exact Solution to Our Problem

- The state equation formulation of our system

is

$$\frac{dx}{dt} = \dot{x} = -\alpha x$$

This is a linear differential equation with constant coefficients – a type of system that can be solved exactly.

# Solution Procedure

$$\frac{dx}{dt} = -\alpha x$$

- Suppose we start  $x$  at time 0 with initial value  $x(0)$ , and we want to find the value of  $x$  at time  $T$
- Assuming that  $x$  does not start at 0, it will never reach exactly 0, so we can divide the left side by it, and multiply the right side by  $dt$

$$\frac{dx}{x} = -\alpha dt$$

- Integrating both sides

$$\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha dt$$

# Completion of Derivation

$$\int_{t=0}^{t=T} \frac{dx}{x} = \int_{t=0}^{t=T} -\alpha dt = -\alpha \int_{t=0}^{t=T} dt$$

$$\ln x \Big|_{t=0}^{t=T} = -\alpha t \Big|_{t=0}^{t=T}$$

$$\ln x(T) - \ln x(0) = -\alpha T$$

$$\ln x(T) = \ln x(0) - \alpha T$$

$$x(T) = e^{\ln x(0) - \alpha T} = e^{\ln x(0)} e^{-\alpha T} = x(0) e^{-\alpha T}$$

So the stock  $x$  declines as a negative exponential in time  $T$   
i.e. # of people remaining in the stock goes down exponentially w/time

# Fraction of Original People Still in Stock or Who have Left

- Assuming no inflows, the fraction of people still in the stock at time  $T$  is just

(# of people in the stock at time  $T$ )/(initial # of people in the stock)=

$$\frac{x(T)}{x(0)} = \frac{x(0)e^{-\alpha T}}{x(0)} = e^{-\alpha T}$$

- Given that people either stay in the stock or leave, the fraction that have left by time  $T$ =

$$1 - \frac{x(T)}{x(0)} = 1 - e^{-\alpha T}$$

# At Time=1

- At time  $t=1$ , we have a fraction  $e^{-\alpha \cdot 1} = e^{-\alpha}$  in the stock, and a fraction  $1 - e^{-\alpha}$  who have left

- Note: By its Taylor Expansion

$$e^{-\alpha t} = \sum_{i=0}^{\infty} \frac{(-\alpha t)^i}{i!} = 1 + (-\alpha t) + \frac{(-\alpha t)^2}{2!} + \frac{(-\alpha t)^3}{3!} + \dots$$

$$= 1 - \alpha t + \frac{(\alpha t)^2}{2} + \dots$$

- For small  $\alpha t$ , the higher order terms are very small, and this will be approximately  $1 - \alpha t$
- So by time 1 for small  $\alpha$ , approx  $1 - \alpha$  will remain after, and a fraction of  $\alpha$  will have departed

# Mean Time to Transition

- People are leaving via the flow
- Suppose we wish to determine the mean (average) time for a given person in the stock to leave
- Recall: A mean for a continuous probability distribution  $p(t)$  is given by  $\int_{t=-\infty}^{\infty} tp(t)dt$
- Since  $p(t)dt$  is the probability that will leave between  $t$  and  $t+dt$ , this is just the continuous version of

$$E[q(a)] = \sum_{a \in \{\text{Possible values of } a\}} aq(a)$$



# Mean Time to Leave

- $p(t)dt$  here is the likelihood of a person leaving exactly between time  $t$  &  $dt+t$ 
  - We start the simulation at  $t=0$ , so  $p(t)=0$  for  $t<0$
  - For  $t>0$ ,  $P(\text{leaving exactly between time } t \text{ and } dt+t) = P(\text{leaving exactly between time } t \text{ and } t+dt | \text{Still have not left by time } t)P(\text{Still have not left by time } t)$

For  $T>0$ ,  $P(\text{Still have not left by time } t) = e^{-\alpha T}$  (previous slid)

For  $P(\text{leaving exactly between time } t \text{ and } t+dt | \text{Still have not left by time } t)$

Recall: For us, probability of leaving in a time  $dt$  always  $=\alpha dt$  (remember:  $\alpha$  is prob of leaving/unit time)

Thus  $P(\text{leaving exactly between time } t \text{ and } t+dt | \text{Still have not left by time } t) = \alpha dt$

$P(t)dt = P(\text{leaving exact b.t. time } t \text{ & } dt+t) = (e^{-\alpha T})(\alpha dt) = \alpha e^{-\alpha T} dt$

# Derivation of Mean

- $P(t)dt = P(\text{leaving exactly between time } t \text{ \& } dt+t) = (e^{-\alpha T})(\alpha dt) = \alpha e^{-\alpha T} dt$
- Now that we have found the function  $p(t)$ , we must do the integral  $\int_{t=-\infty}^{t=\infty} tp(t)dt$  to derive the mean
- Here  $E[p(t)] = \int_{t=-\infty}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} tp(t)dt = \int_{t=0}^{t=\infty} t\alpha e^{-\alpha T} dt = \alpha \int_{t=0}^{t=\infty} te^{-\alpha T} dt$

# Recall: Integration by Parts

- We have  $E[p(t)] = \alpha \int_{t=0}^{t=\infty} t e^{-\alpha T} dt = \alpha \left( \int_{t=0}^{t=\infty} t e^{-\alpha T} dt \right)$
- To solve the term in brackets, we will use integration by parts
- Integration by parts exploits the following/!

$$\frac{d(uv)}{dt} = u \frac{dv}{dt} + v \frac{du}{dt}$$

$$d(uv) = u dv + v du$$

$$\int d(uv) = \int u dv + \int v du$$

$$uv = \int u dv + \int v du$$

and thus

$$\int u dv = uv - \int v du$$

# Recall: Integration by Parts

- To solve  $\int_{t=0}^{t=\infty} te^{-\alpha T} dt$  we will use integration by parts

$$u = t \Rightarrow du = \frac{du}{dt} dt = 1 dt = dt$$

- Here

$$dv = e^{-\alpha T} dt \Rightarrow v = \int e^{-\alpha T} dt = \frac{-e^{-\alpha T}}{\alpha}$$

- From the previous page, we know

$$\int_{t=0}^{t=\infty} te^{-\alpha T} dt = \int_{t=0}^{t=\infty} u dv = uv - \int_{t=0}^{t=\infty} v du = \left( t \frac{-e^{-\alpha T}}{\alpha} \right) \Big|_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty} \left( \frac{-e^{-\alpha T}}{\alpha} \right) dt$$

$$= \left( \frac{-te^{-\alpha T}}{\alpha} \right) \Big|_{t=0}^{t=\infty} + \frac{1}{\alpha} \int_{t=0}^{t=\infty} e^{-\alpha T} dt = (0 - 0) + \frac{1}{\alpha} \left( \frac{-1}{\alpha} e^{-\alpha T} \right) \Big|_{t=0}^{t=\infty} =$$

$$\frac{1}{\alpha} \left( 0 - \frac{-1}{\alpha} \right) = \frac{1}{\alpha^2}$$

# Thus

- The mean time (the *delay associated with a first order delay*) is thus given by

$$E[p(t)] = \alpha \int_{t=0}^{t=\infty} t e^{-\alpha T} dt = \alpha \left( \int_{t=0}^{t=\infty} t e^{-\alpha T} dt \right)$$
$$= \alpha \left( \frac{1}{\alpha^2} \right) = \frac{1}{\alpha}$$

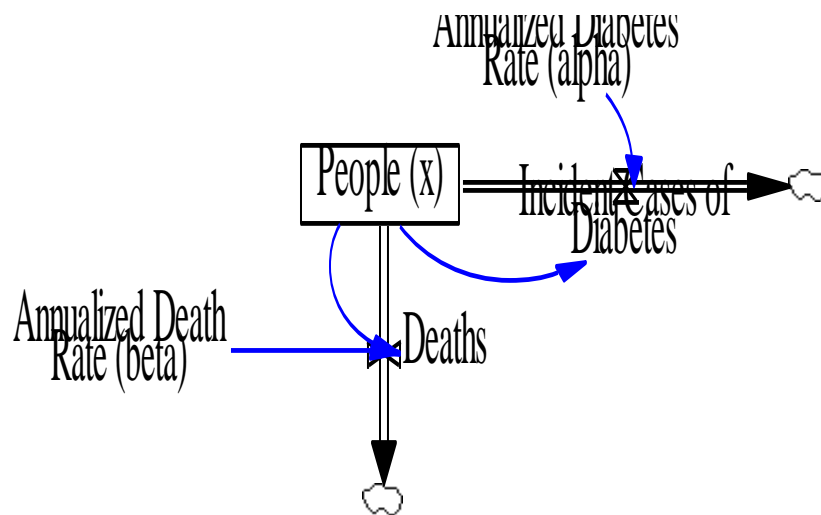
- So e.g. if we have an annualized rate of diabetes incidence, the mean time to develop diabetes (independent of other risks) is just the reciprocal of that rate (i.e. 1 over that rate)

# Computer Exercise: Simulating a First Order Delay

- Create a first order delay
- Feed in a “step function” that rises suddenly at time 10.
- How does the output from the stock change over time?

# Competing Risks

- Suppose we have another outflow from the stock. How does that change our mean time of proceeding specifically down flow 1 (here, developing diabetes)?



# Competing Risks Stock Trajectory

## Solution Procedure

$$\frac{dx}{dt} = -\alpha x - \beta x = -(\alpha + \beta)x$$

- Suppose we start  $x$  at time 0 with initial value  $x(0)$ , and we want to find the value of  $x$  at time  $T$
- This is just like our previous differential equation, except that “ $\alpha$ ” has been replaced by “ $(\alpha + \beta)$ ”
  - The solution must therefore be the same as before, with the appropriate replacement
  - Thus

$$x(T) = x(0)e^{-(\alpha + \beta)T}$$



# Mean Time to Leave: Competing Risks

- $p(t)dt$  here is the likelihood of a person leaving *via flow 1* (e.g. developing T2DM) exactly between time  $t$  &  $t+dt$ 
  - We start the simulation at  $t=0$ , so  $p(t)=0$  for  $t<0$
  - For  $t>0$ ,  $P(\text{leaving on flow 1 exactly between time } t \text{ \& } t+dt) = P(\text{leaving on flow 1 exactly between time } t \text{ \& } t+dt \mid \text{Still have not left by time } t)P(\text{Still have not left by time } t)$

For  $T>0$ ,  $P(\text{Still have not left by time } T) = e^{-(\alpha+\beta)T}$

For  $P(\text{leaving exactly between time } t \text{ and } t+dt \mid \text{Still have not left by time } t)$

Recall: For us, probability of leaving in a time  $dt$  always  $= \alpha dt$

Thus  $P(\text{leaving exactly between time } t \text{ and } t+dt \mid \text{Still have not left by time } t) = \alpha dt$

$$P(t)dt = P(\text{leaving exact b.t. time } t \text{ \& } t+dt) = \alpha e^{-(\alpha+\beta)T} dt$$

# Mean Time to Transition via Flow 1: Competing Risks

- By the same procedure as before, we have

$$E[p(t)] = \alpha \int_{t=0}^{t=\infty} t e^{-(\alpha+\beta)T} dt$$

- Using the formula we derived for the integral expression, we have

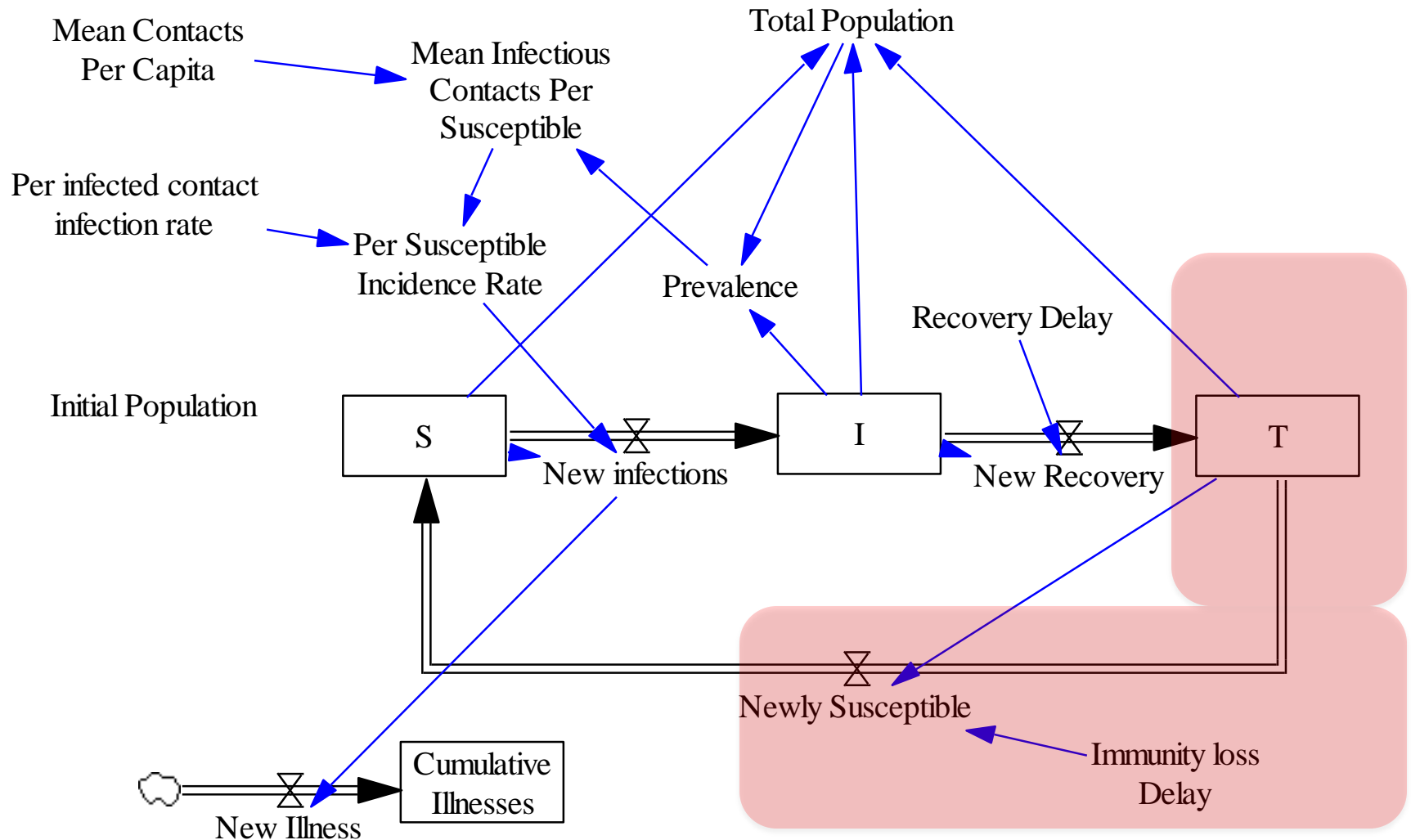
$$E[p(t)] = \frac{\alpha}{(\alpha + \beta)^2}$$

- [Forward reference: We can guess this based on dimensional considerations & symmetry]
- Note that this correctly approaches the single-flow case as  $\beta \rightarrow 0$

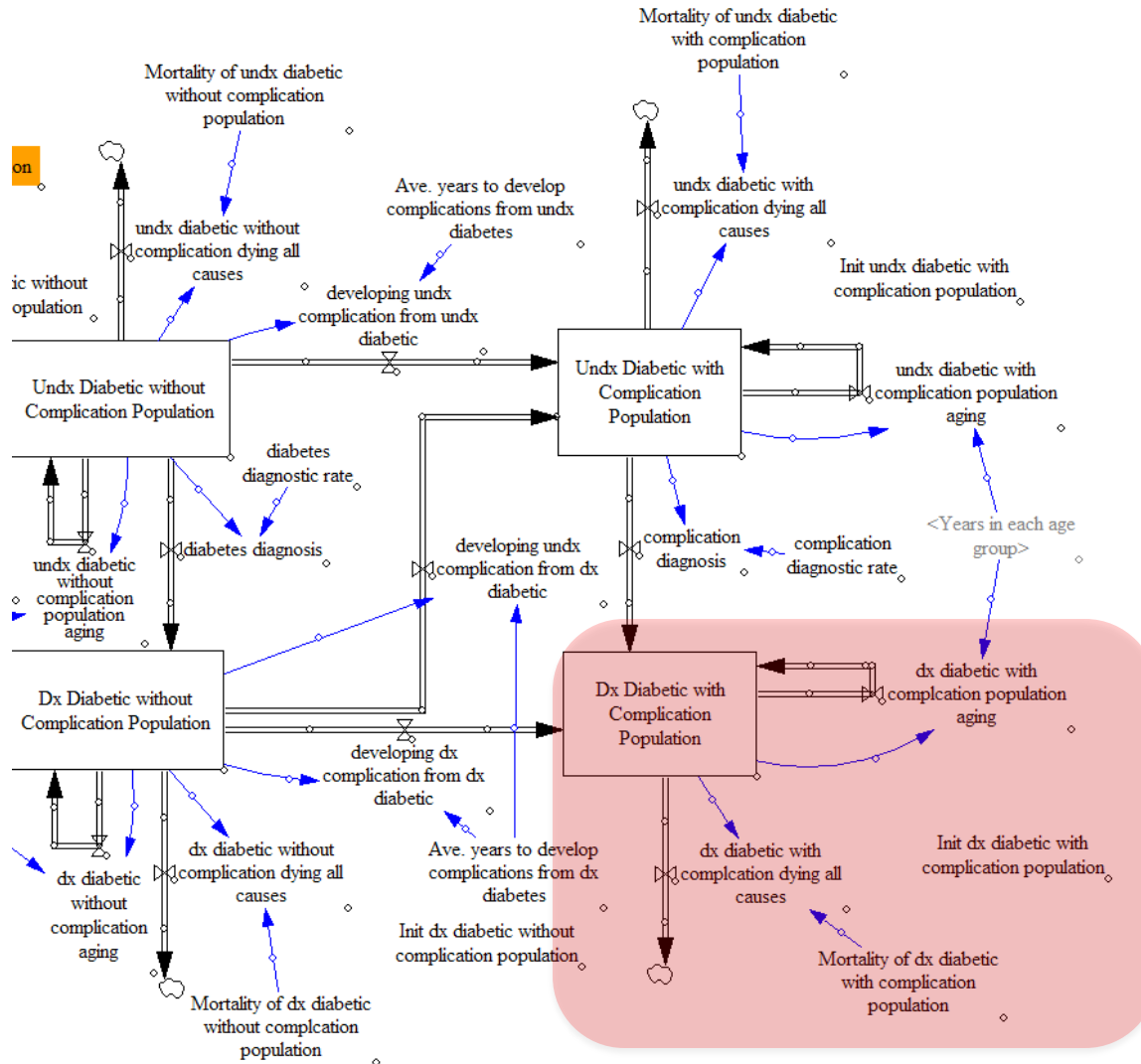
# Computer Exercise: Simulating a First Order Delay

- Create a first order delay
- Feed in a “step function” that rises suddenly from 0 to 20 at time 10
  - Use formula if then else( $\text{Time} > 10, 20, 0$ )
- Questions to ponder
  - How does the output from the stock change over time?
  - How does the equilibrium value of the stock vary with chance of proceeding (alpha)?

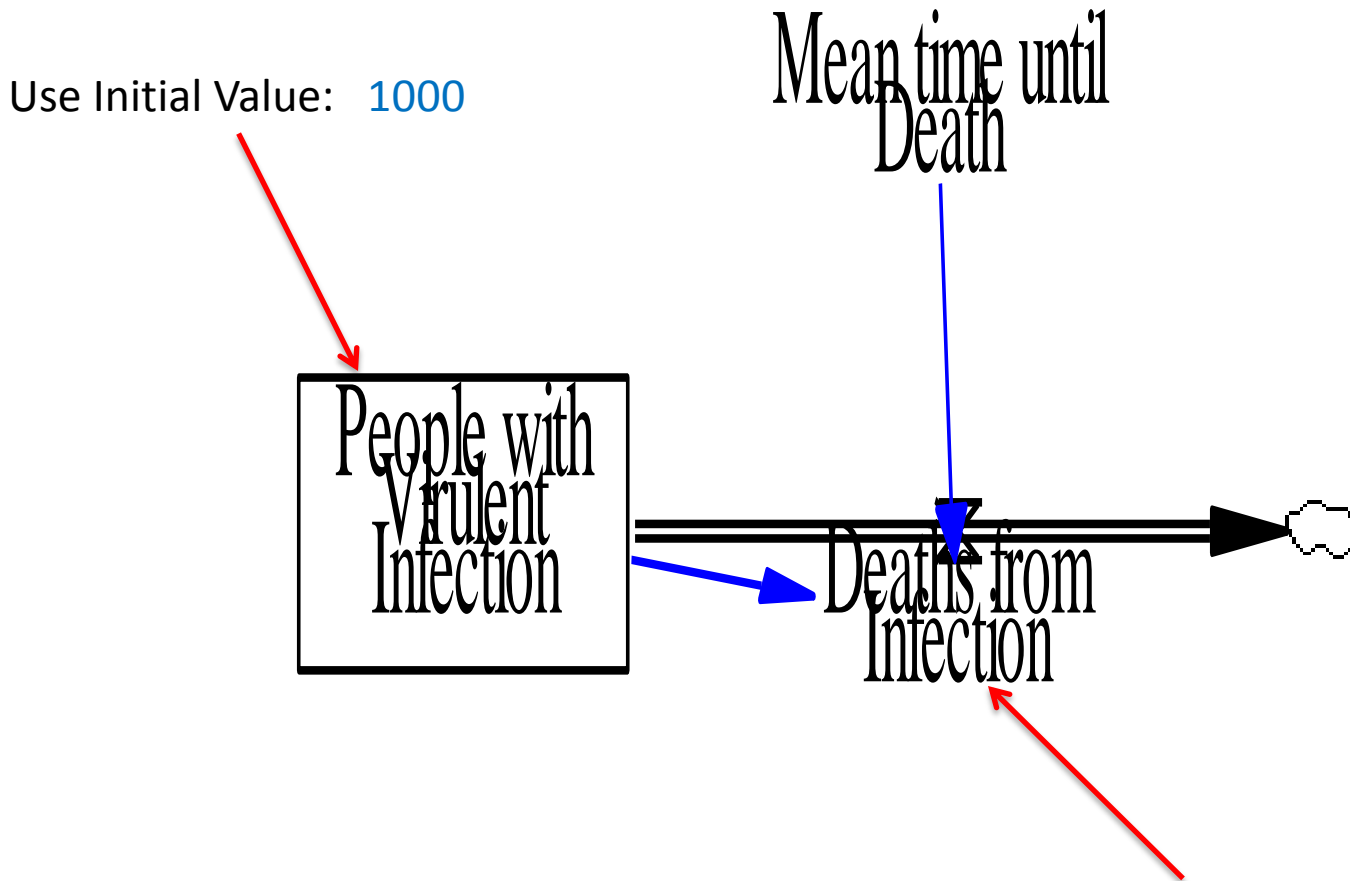
# First Order Delays in Action: Simple SIT Model



# First Order Delays in Action: Simple SIT Model



# Recall: Simple First-Order Decay



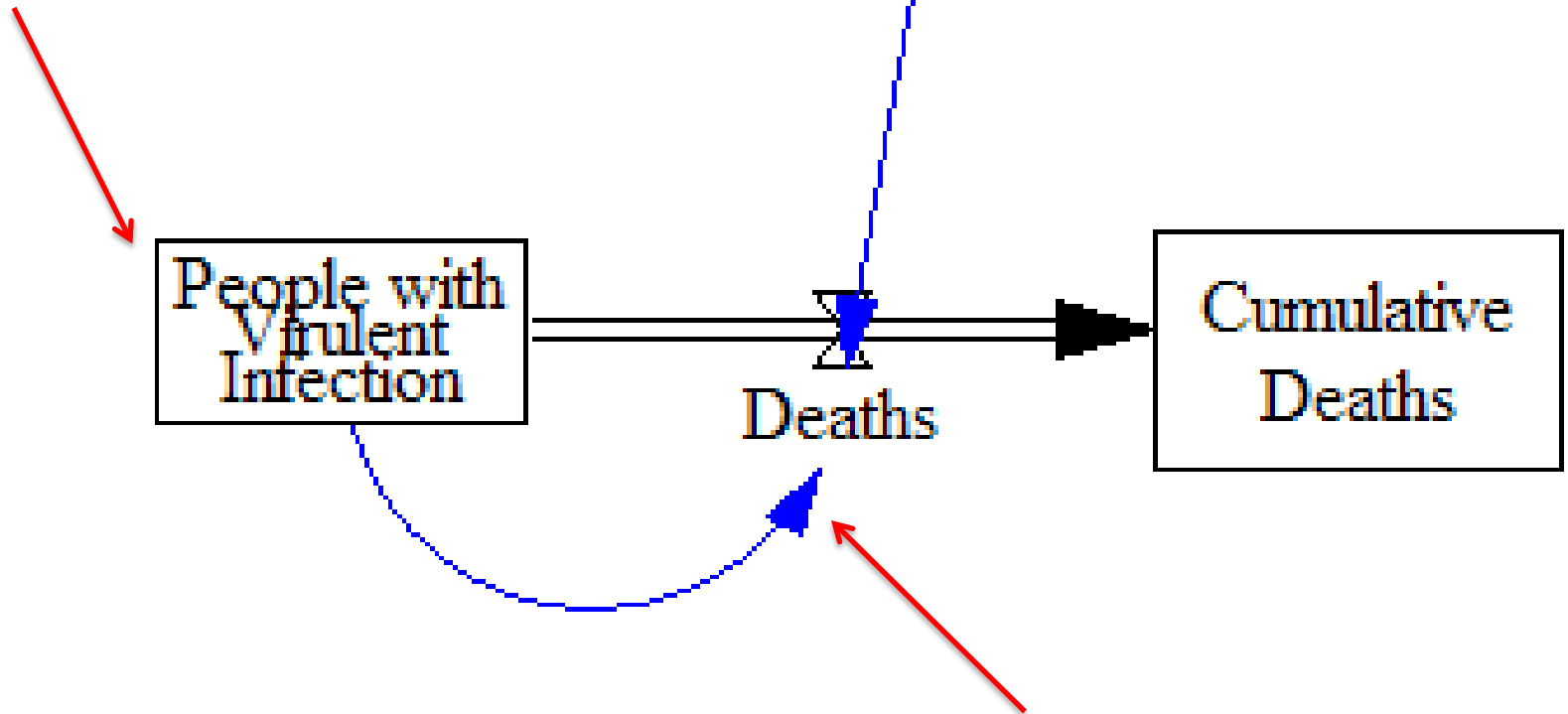
Use Formula:  $\text{People with Virulent Infection} / \text{Mean time until Death}$

# First-Order Decay (Variant of Last Time)

Recall: How does this relate to the mean time until death?

Use Initial Value: 1000

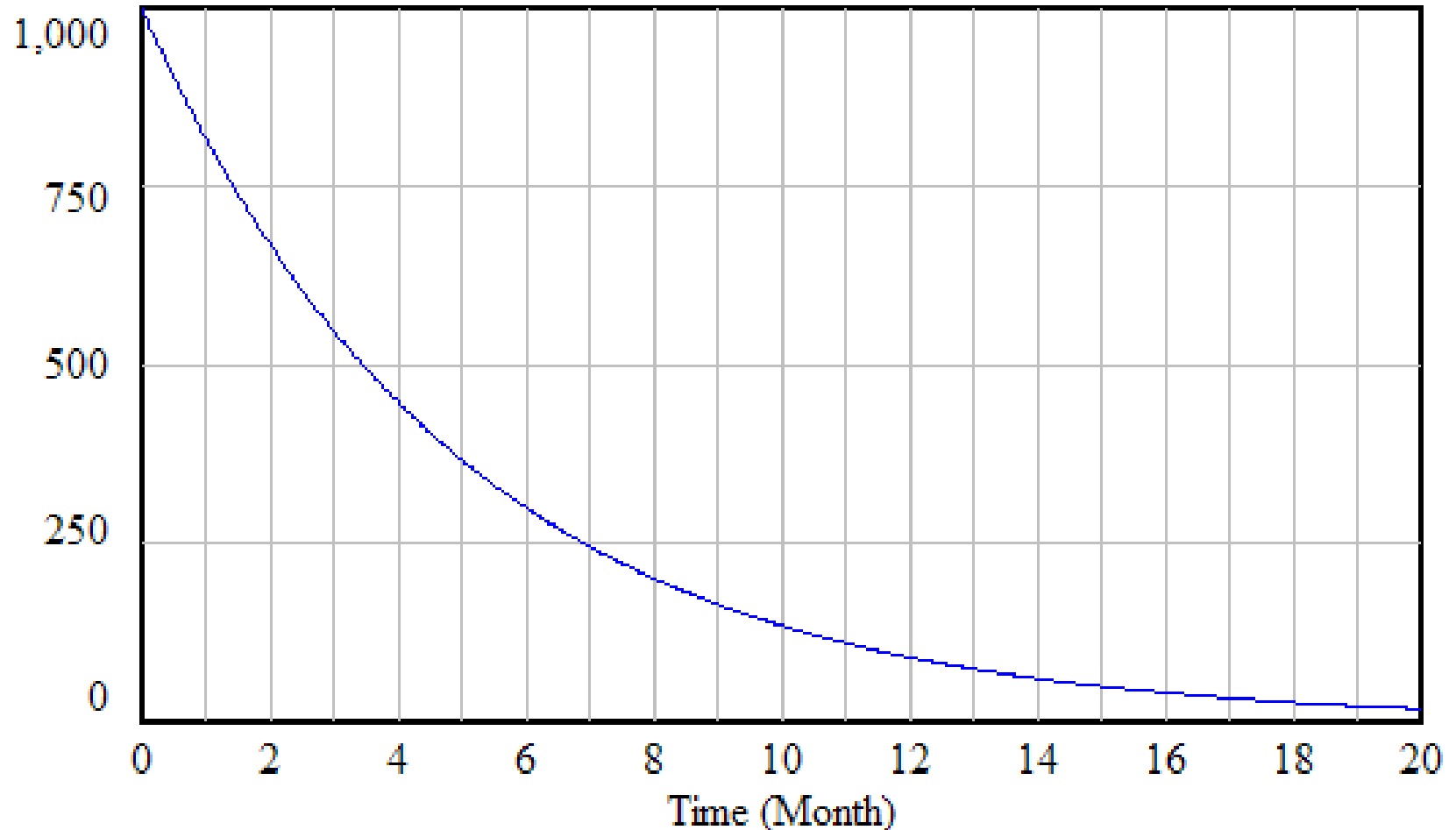
Per Month Use Value: 0.2  
Likelihood of Death



Use Formula:  $\text{People with Virulent Infection} * \text{Per Month Likelihood of Death}$

# People in Stock

People with Virulent Infection



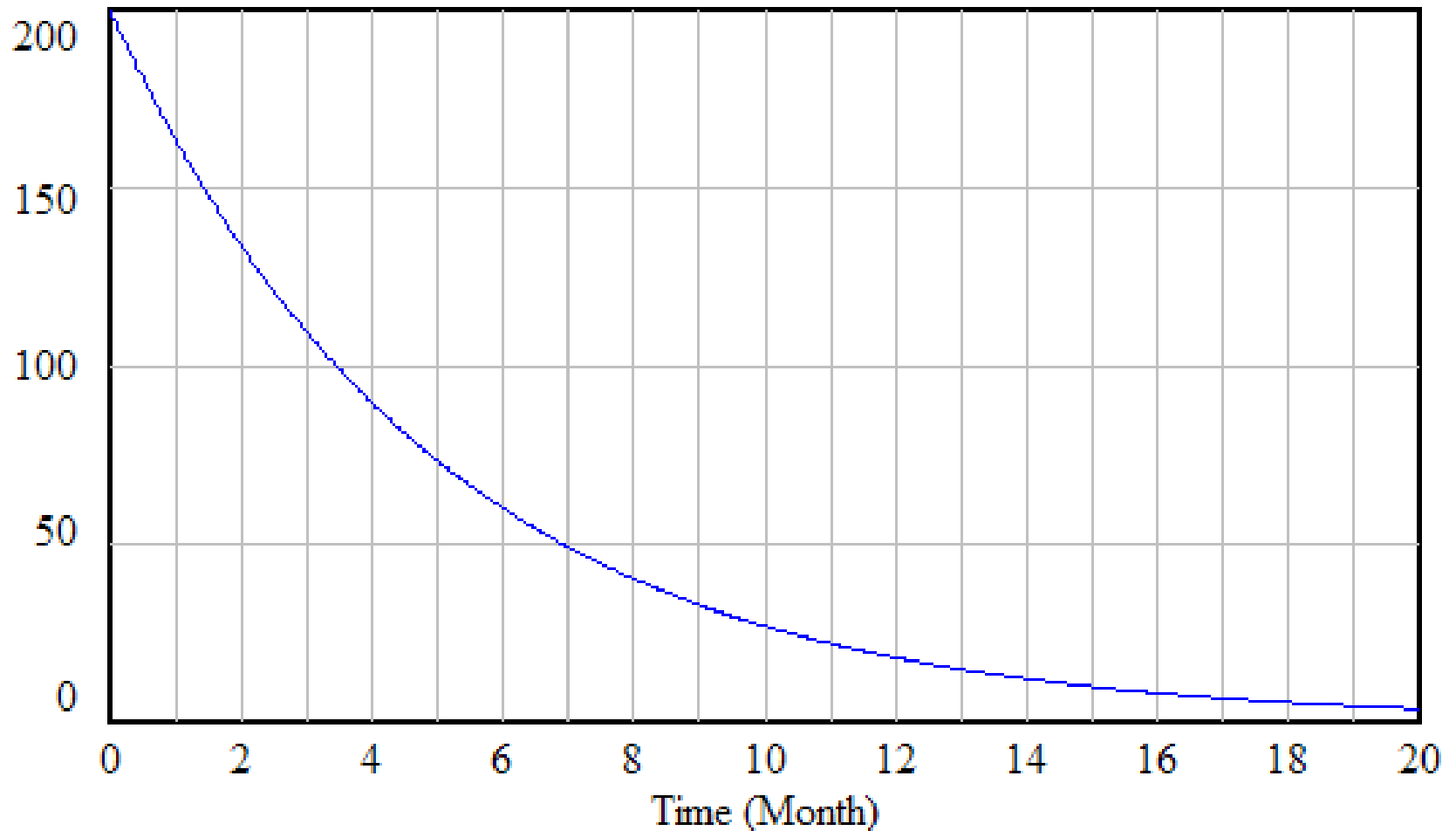
People with Virulent Infection : Baseline





# Flow Rate of Deaths

Deaths

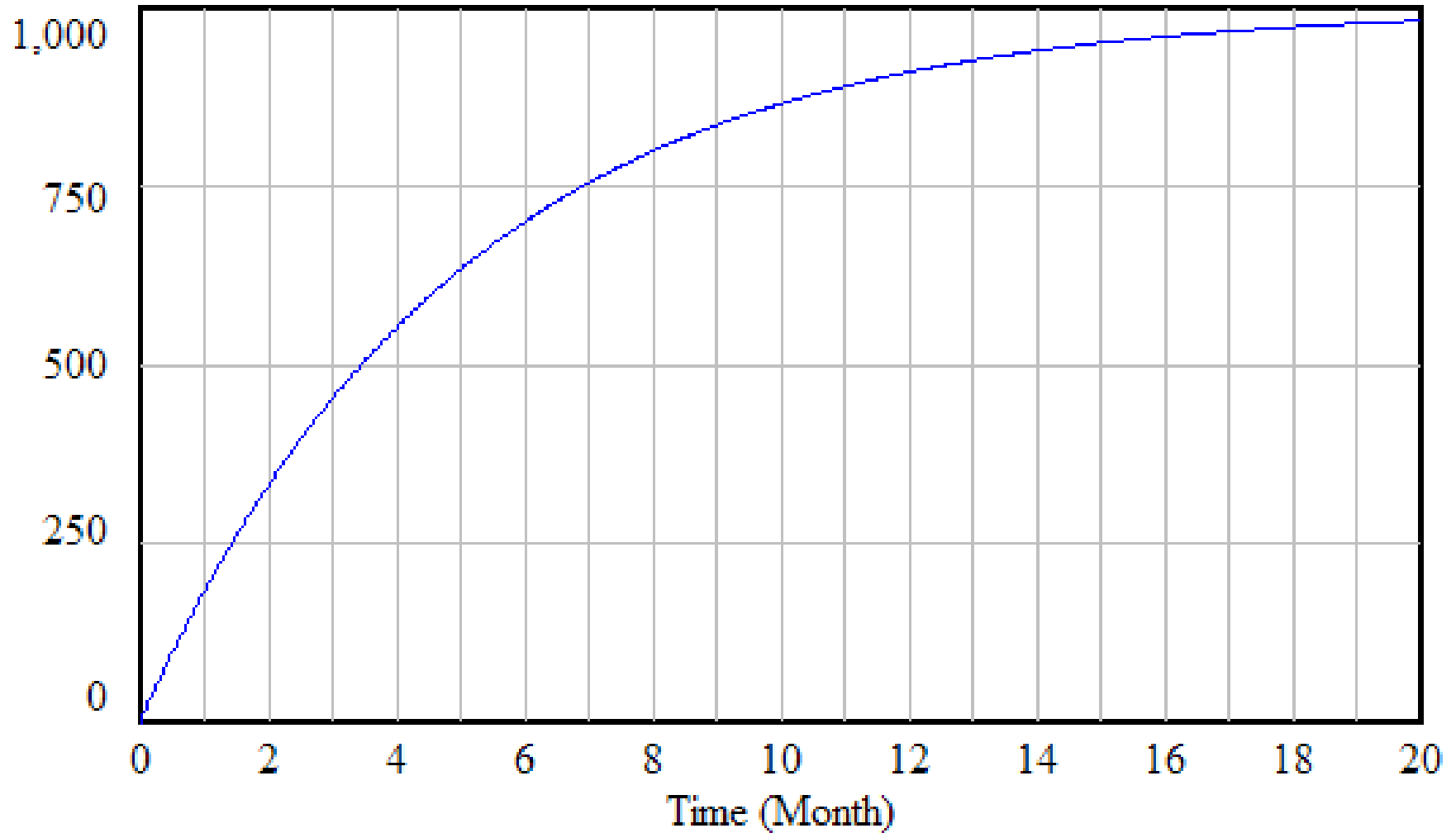


Deaths : Baseline



# Cumulative Deaths

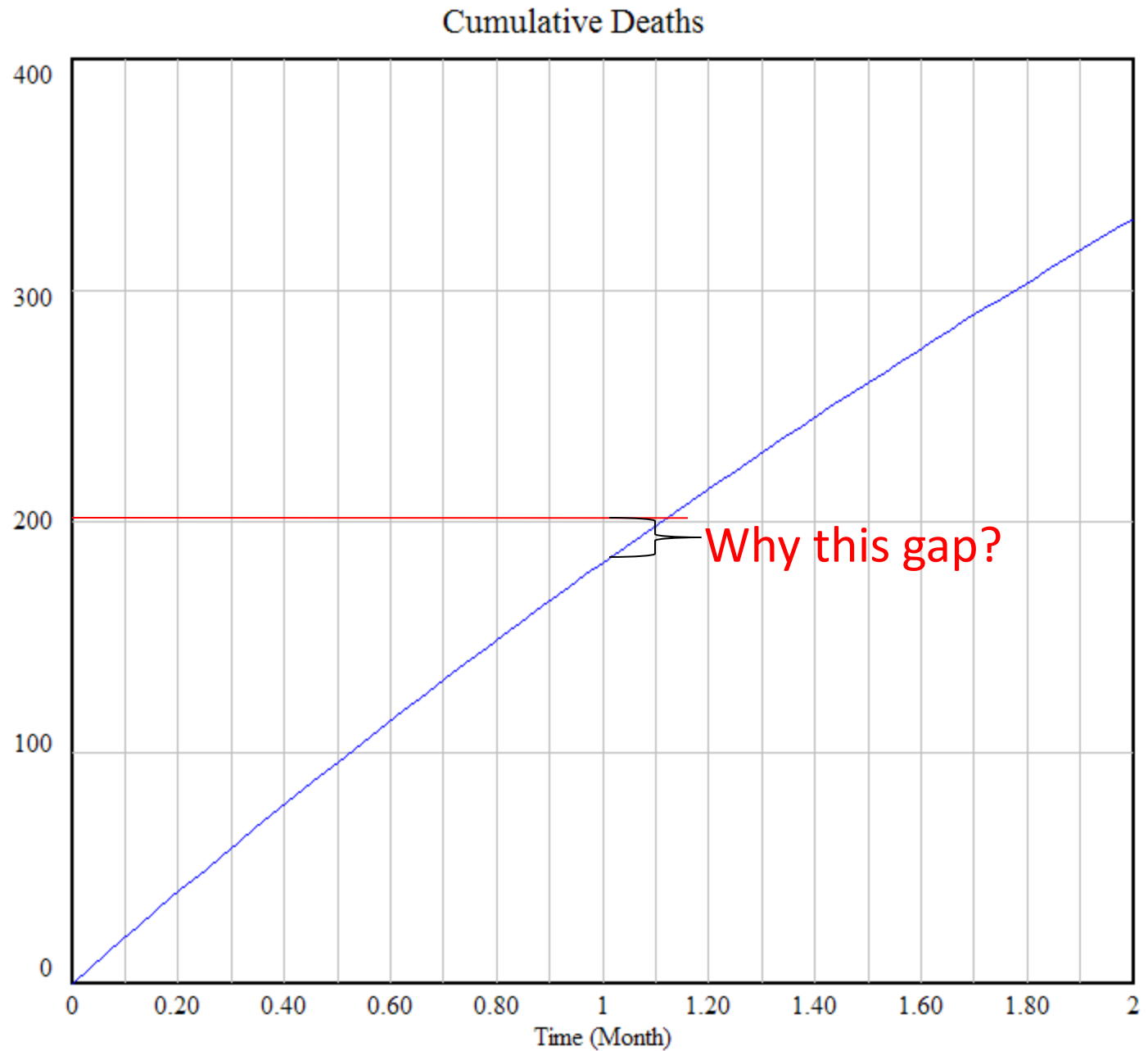
Cumulative Deaths



Cumulative Deaths : Baseline

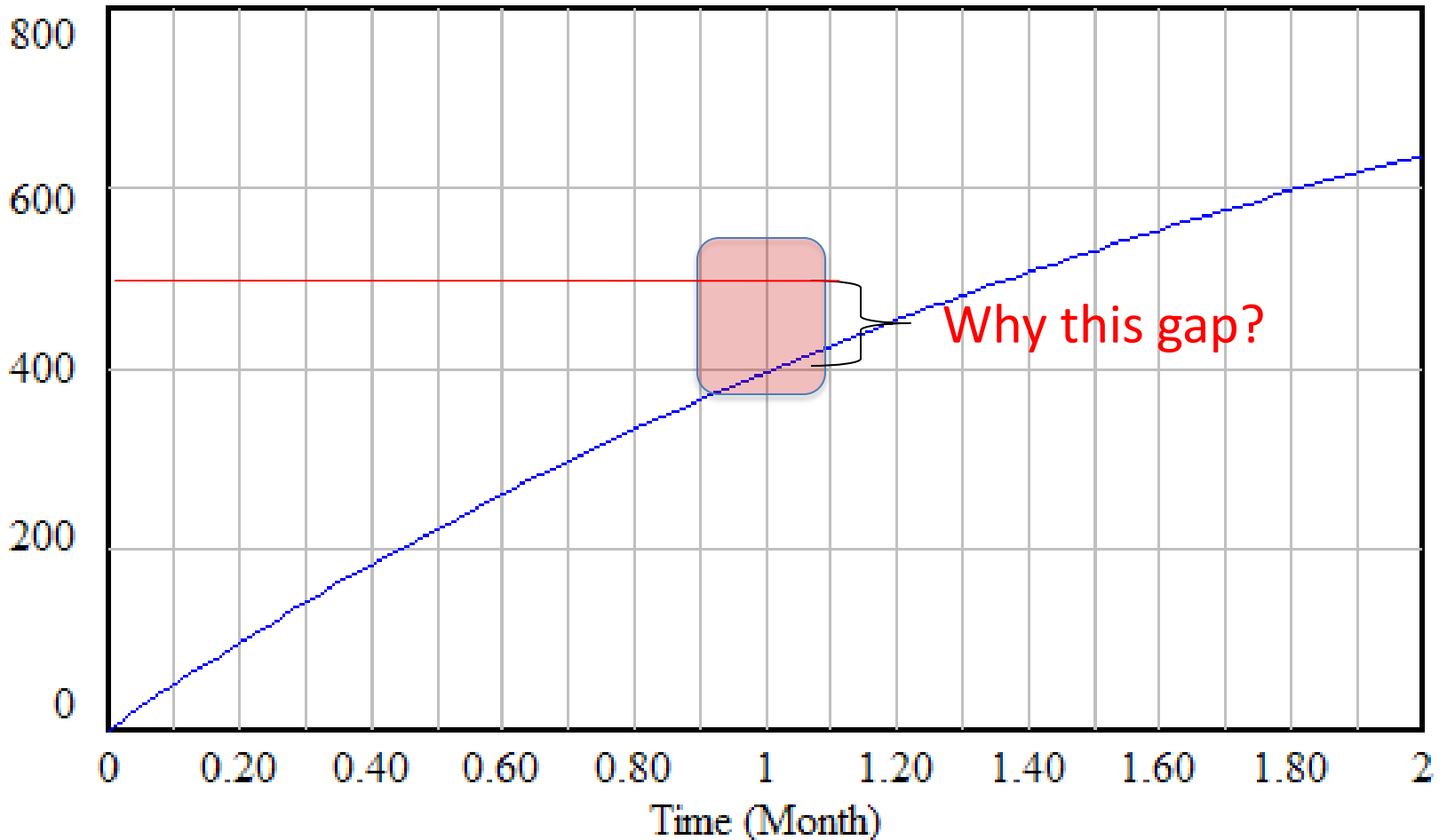


# Closeup



# 50% per Month Risk of Deaths

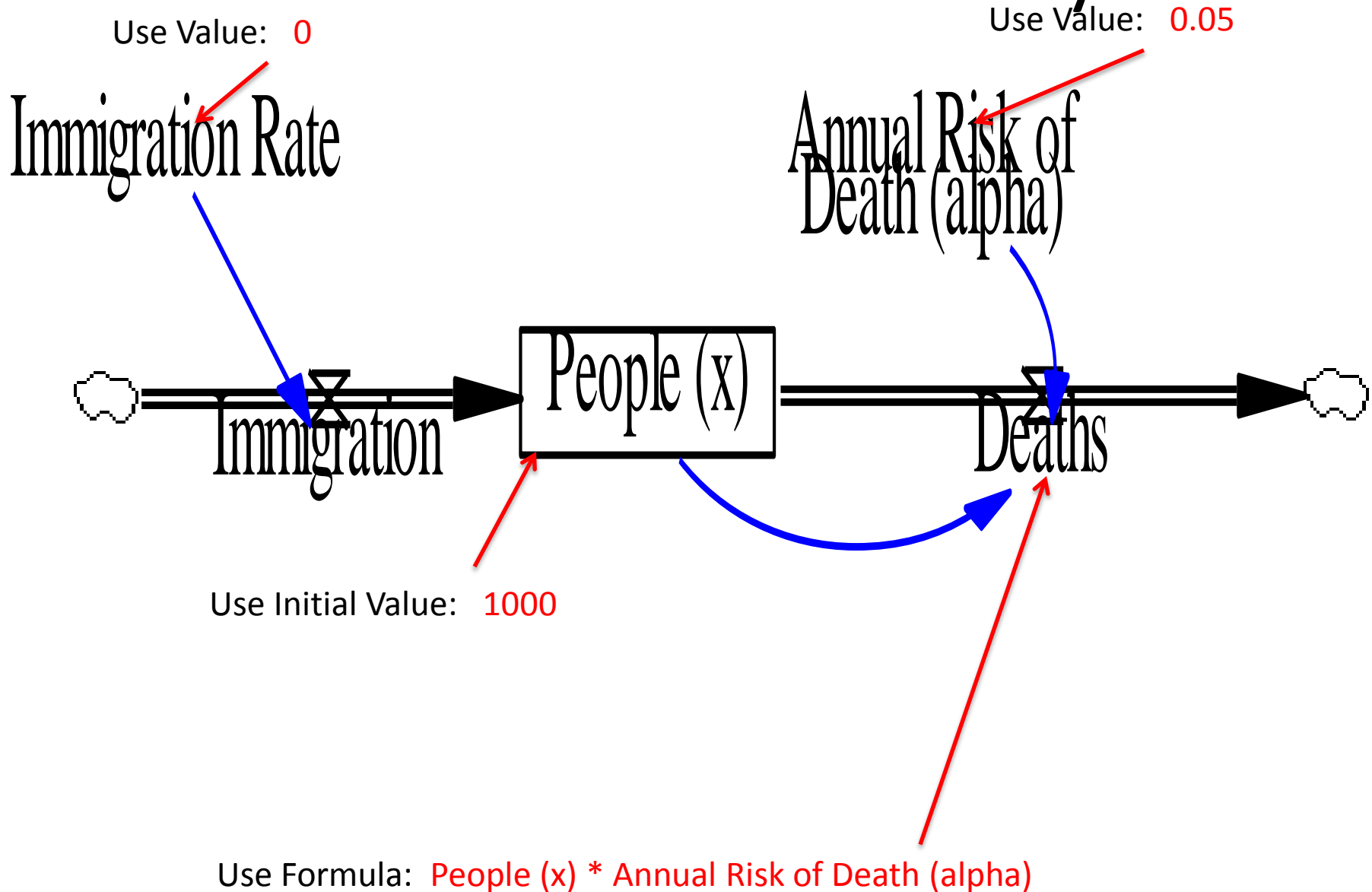
Cumulative Deaths



Answer: The “Gap” is Present Because not all 1000 people are at risk for a month!

- The value of the stock is declining over the first month
- The rate of death indicates that 20% of the population will die per month
- While we may have been expecting 200 people (20% of the 1000) to die, this (erroneously) assumes that all 1000 were at risk for the entire month
  - In fact, because the stock was declining, there were considerably fewer people at risk, meaning that we have fewer deaths
- If we had maintained 1000 people in the stock for the 1<sup>st</sup> month, 1000 people would have died!

# Recall: First Order Delay

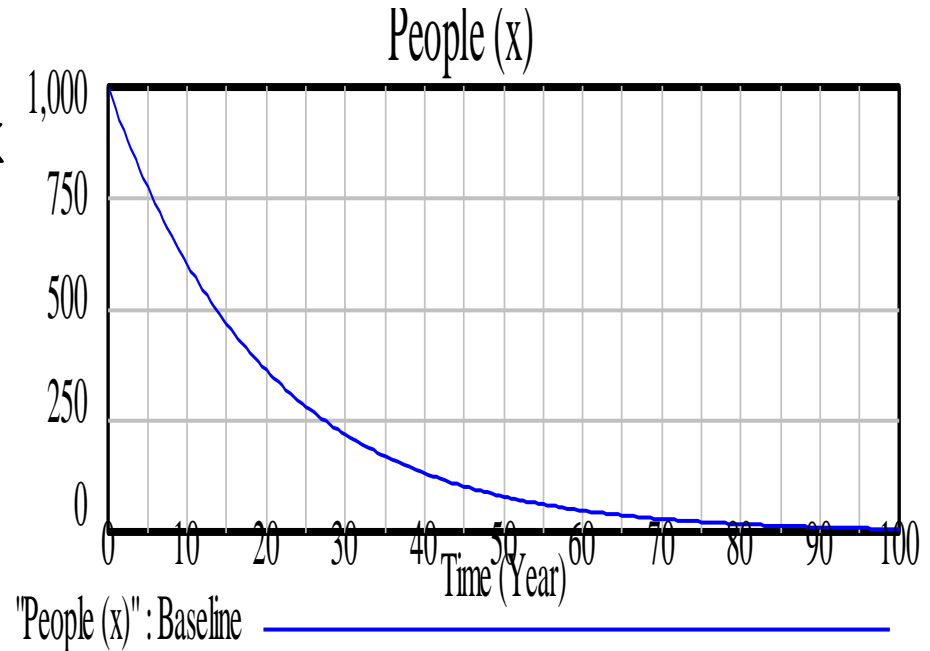


# Questions

- What is behaviour of stock  $x$ ?
- What is the mean time until people die?
- Suppose we had a constant inflow – what is the behaviour then?

# Answers

- Behaviour Of Stock



- Mean Time Until Death

Recall that if coefficient of first order delay is  $\alpha$  , then mean time is  $1/\alpha$  (Here,  $1/0.05 = 20$  years)



# Equilibrium Value of a First-Order Delay

- Suppose we have flow of rate  $i$  into a stock with a first-order delay out
  - This could be from just a single flow, or many flows
- The value of the stock will approach an equilibrium where inflow=outflow

# Equilibrium Value of 1<sup>st</sup> Order Delay

- Recall: Outflow rate for 1<sup>st</sup> order delay= $\alpha x$ 
  - Note that this depends on the value of the stock!
- Inflow rate= $i$
- At equilibrium, the level of the stock must be such that inflow=outflow
  - For our case, we have

$$\alpha x = i$$

$$\text{Thus } x = i / \alpha$$

(equivalently,  $x = i * \text{Mean time to Transition}$ )

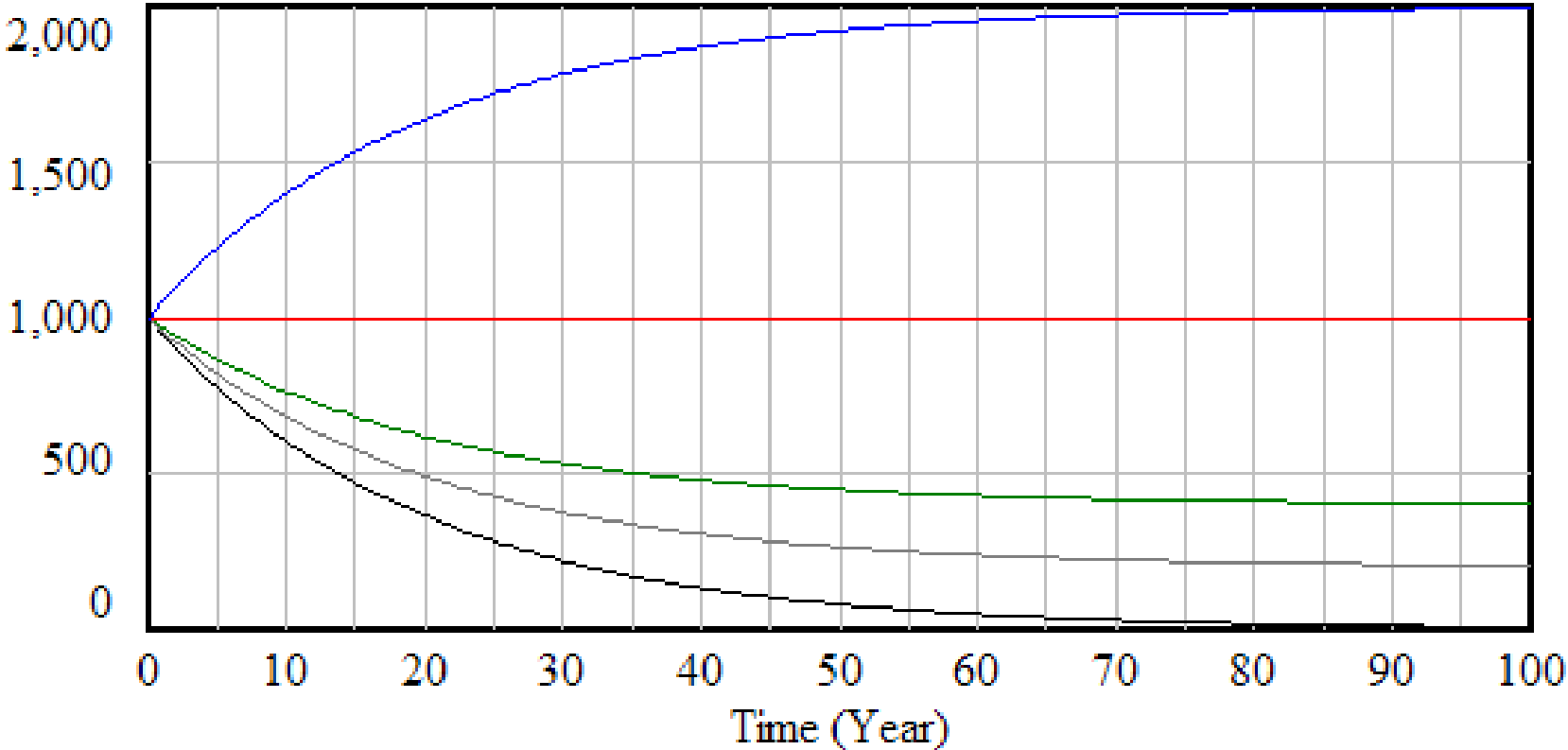
The lower the chance of leaving per time unit (or the longer the delay), the larger the equilibrium value of the stock must be to make outflow=inflow

# Scenarios for First Order Delay: Variation in Inflow Rates

- For different immigration (inflows) (what do you expect?)
  - Inflow=10
  - Inflow=20
  - Inflow=50
  - Inflow=100
  - Why do you see this “goal seeking” pattern?
  - What is the “goal” being sought?

# Behaviour of Stock for Different Inflows

People (x)

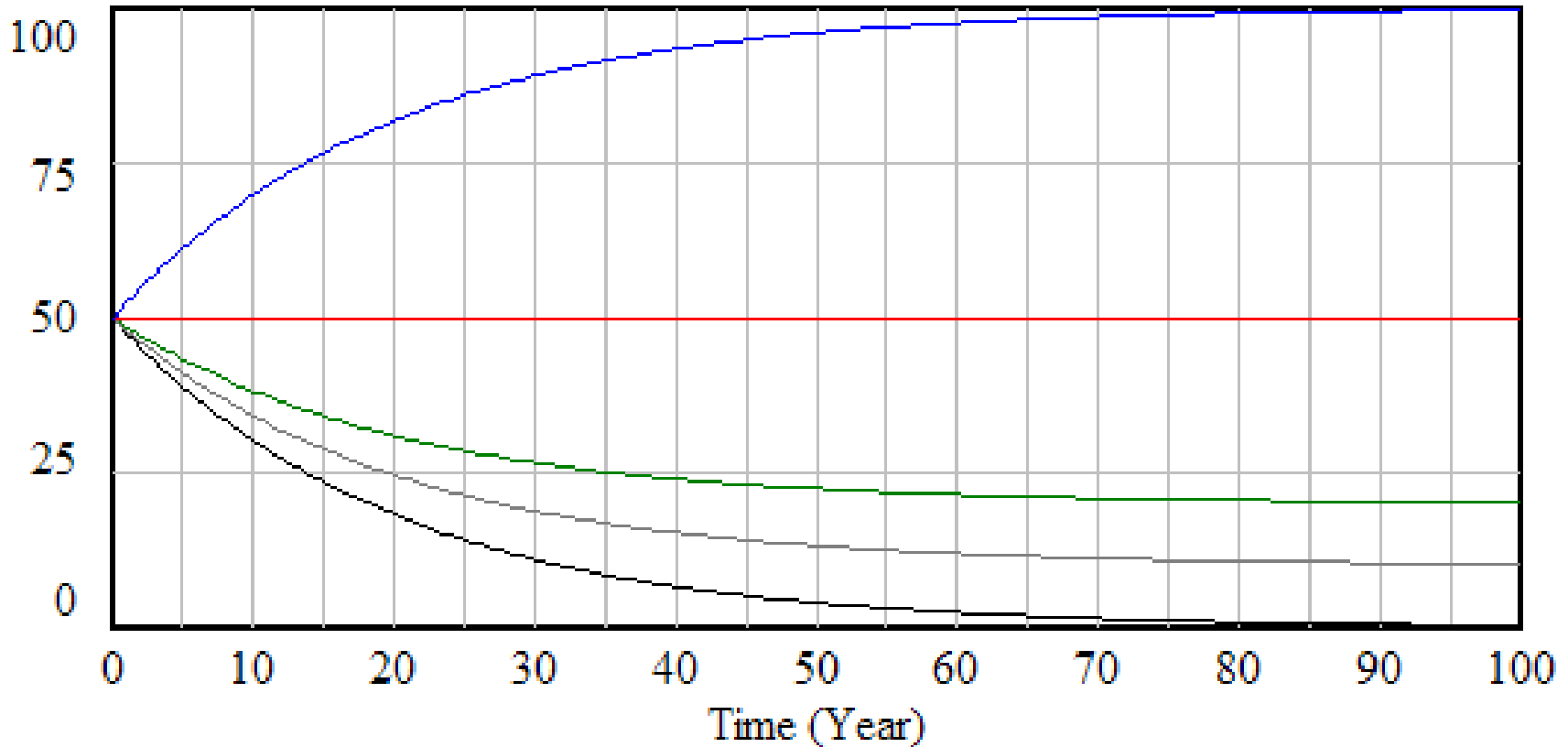


- "People (x)" : Alternative Inflow=100
- "People (x)" : Alternative Inflow=50
- "People (x)" : Alternative Inflow=20
- "People (x)" : Alternative Inflow=10
- "People (x)" : Alternative Inflow=0

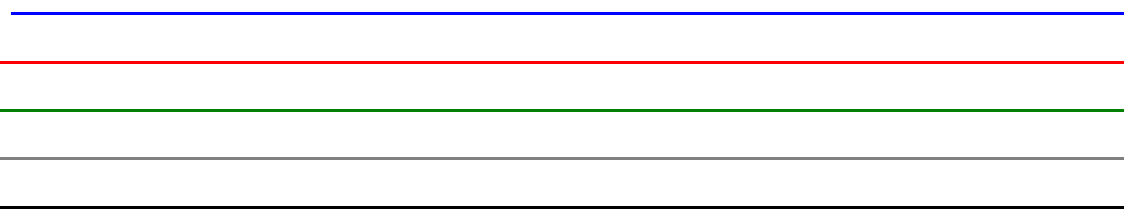
Why do we see this behaviour?

# Behaviour of *Outflow* for Different Inflows

## Deaths



Deaths : Alternative Inflow=100  
Deaths : Alternative Inflow=50  
Deaths : Alternative Inflow=20  
Deaths : Alternative Inflow=10  
Deaths : Alternative Inflow=0

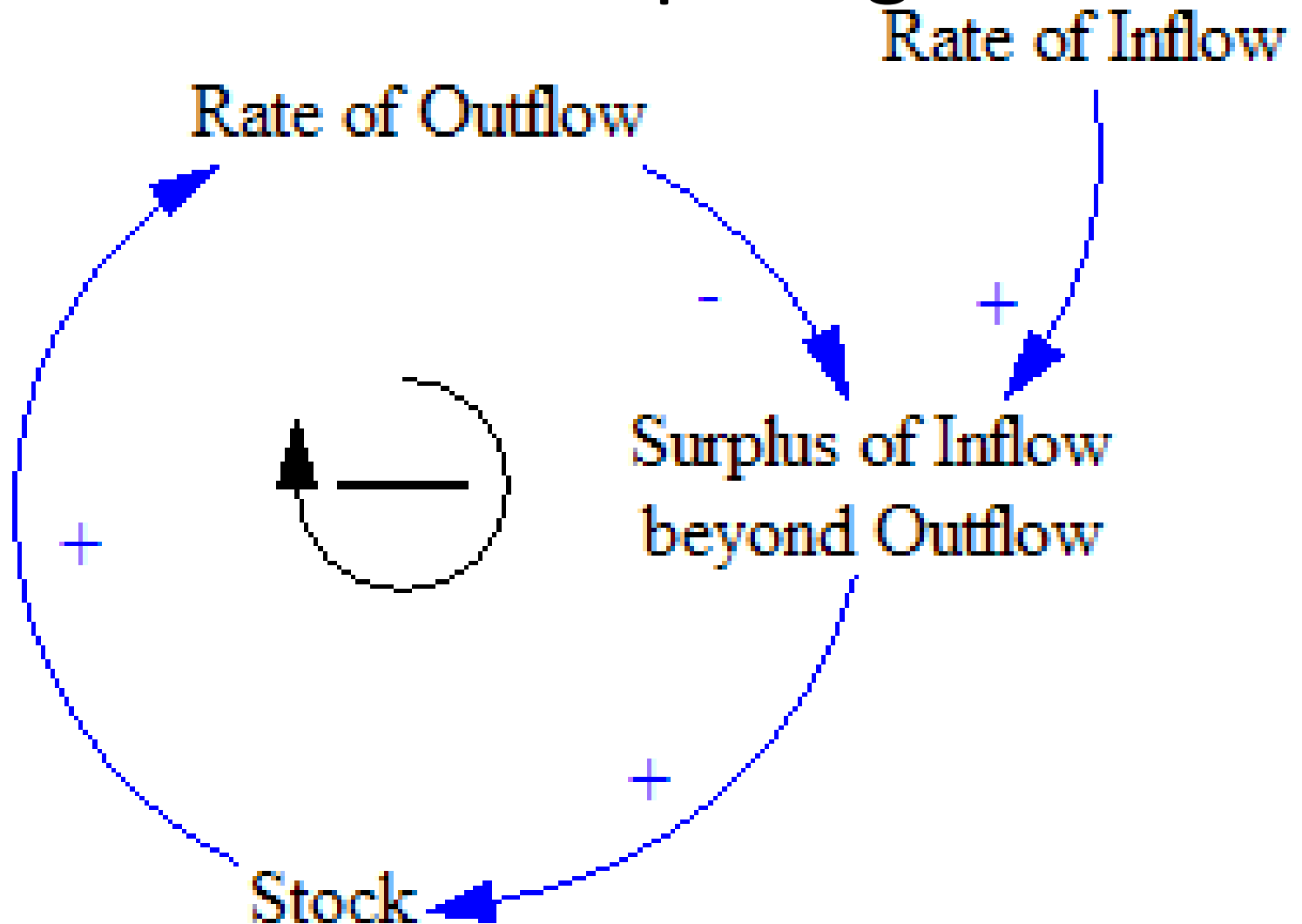


Why do we see this behaviour? Imbalance (gap) causes change to stock (rise or fall)  $\Rightarrow$  change to outflow to lower gap **until outflow=inflow**

# Goal Seeking Behaviour

- The goal seeking behaviour is associated with a negative feedback loop
  - The larger the population in the stock, the more people die per year
- If we have more people coming in than are going out per year, the stock (and, hence, outflow!) rises until the point where inflow=outflows
- If we have fewer people coming in than are going out per year, the stock declines (& outflow) declines until the point where inflow=outflows

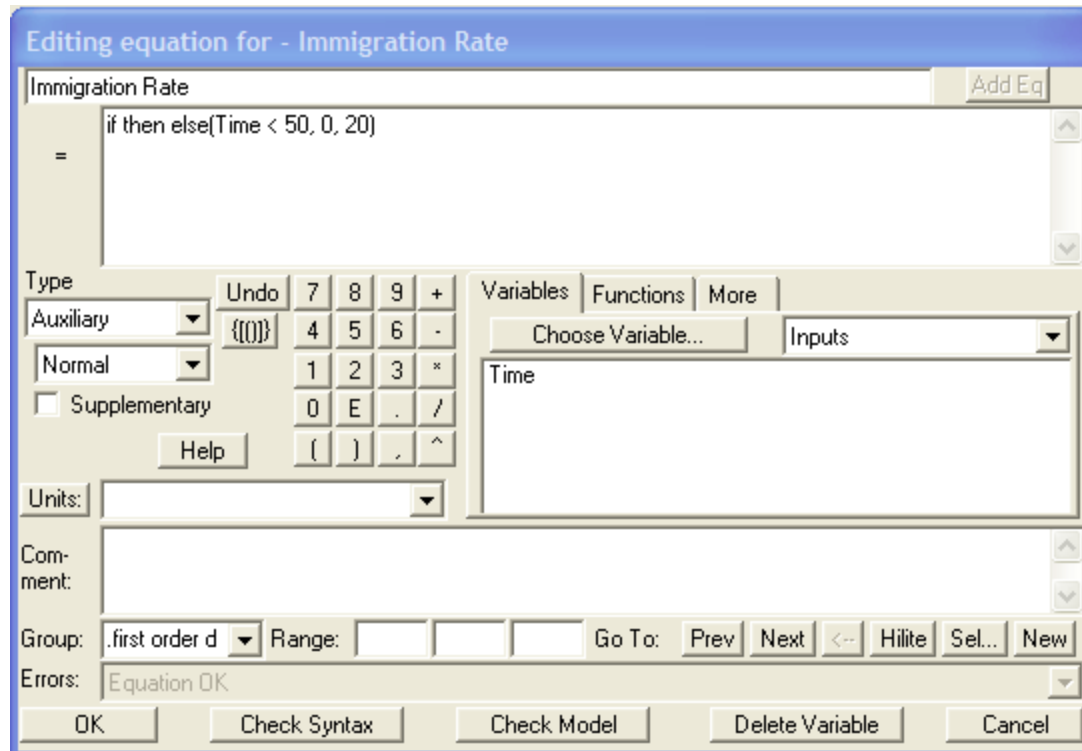
# As a Causal Loop Diagram



What does this tell us about how the system would respond to a sudden change in immigration?

# Response to a Change

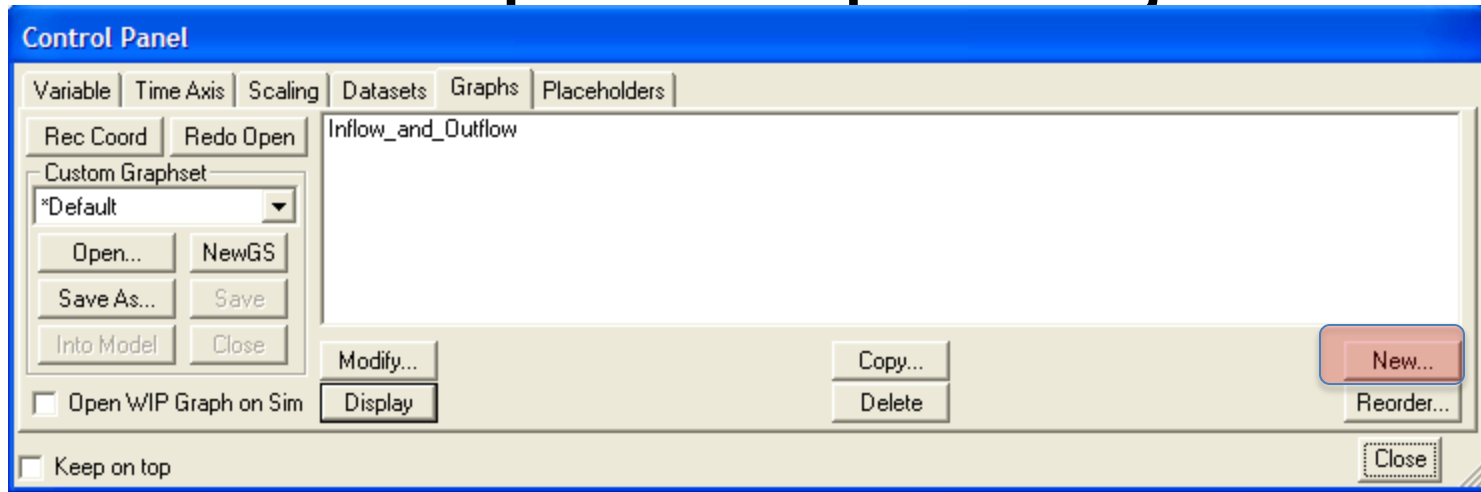
- Feed in an immigration “step function” that rises suddenly from 0 to 20 at time 50



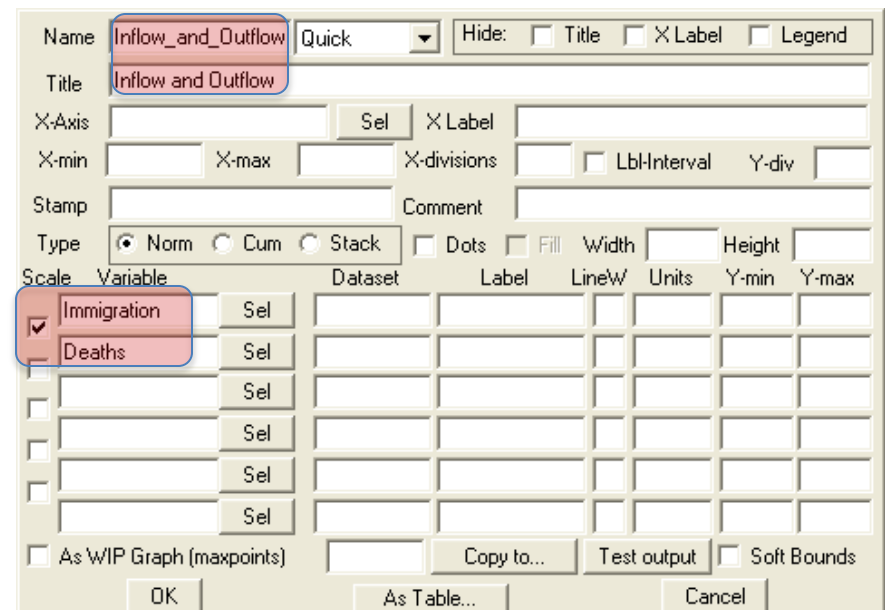
- Set the Initial Value of Stock to 0
- How does the stock change over time?



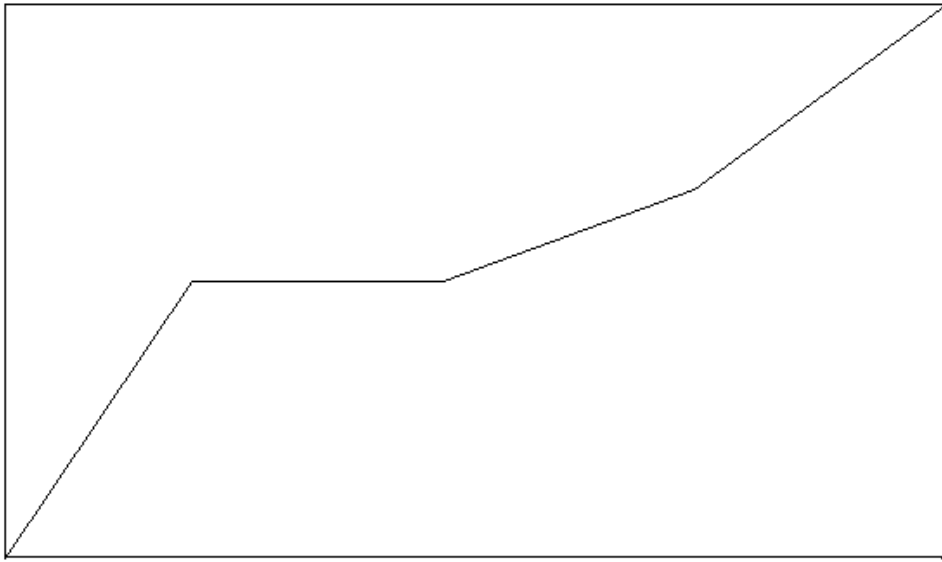
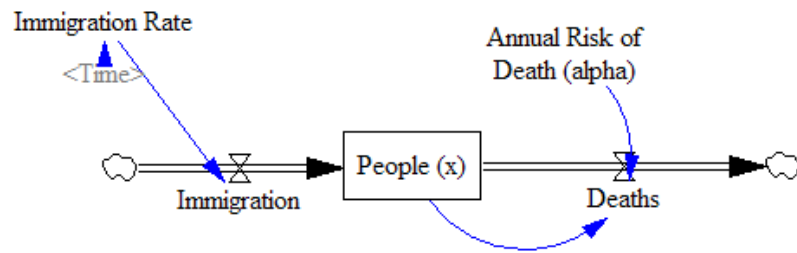
# Create a Custom Graph & Display it as an Input-Output Object



- Editing



# Create Input-Output Object (for Synthesim)



Input Output Object settings

Object Type  
 Input Slider  Output Workbench Tool  Output Custom Graph

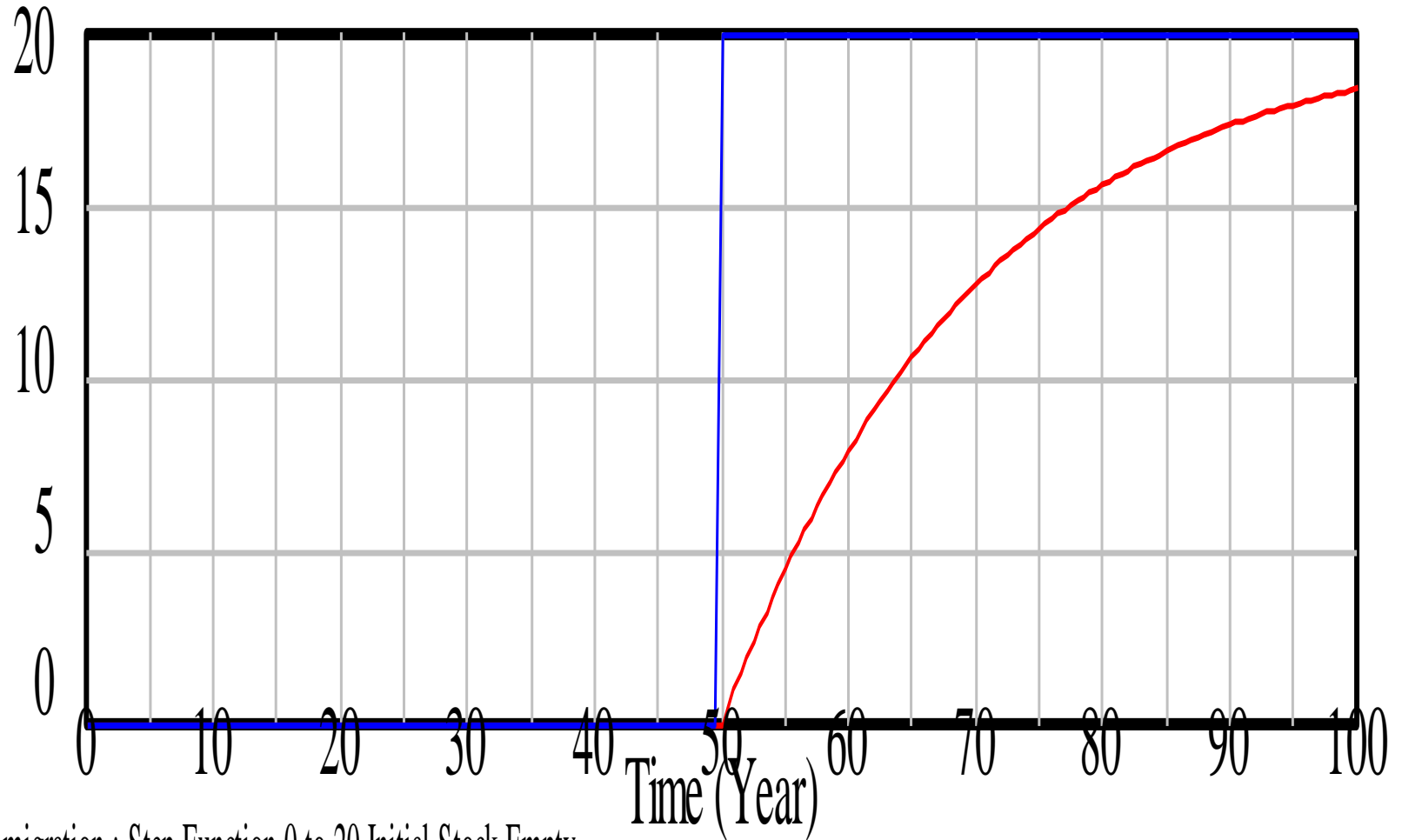
Variable name. Choose:

Slider Settings  
Ranging from  to  with increment   
 Label with varname

Custom Graph or Analysis Tool for Output

# Stock Starting Empty

## *Flow Rates* Inflow and Outflow



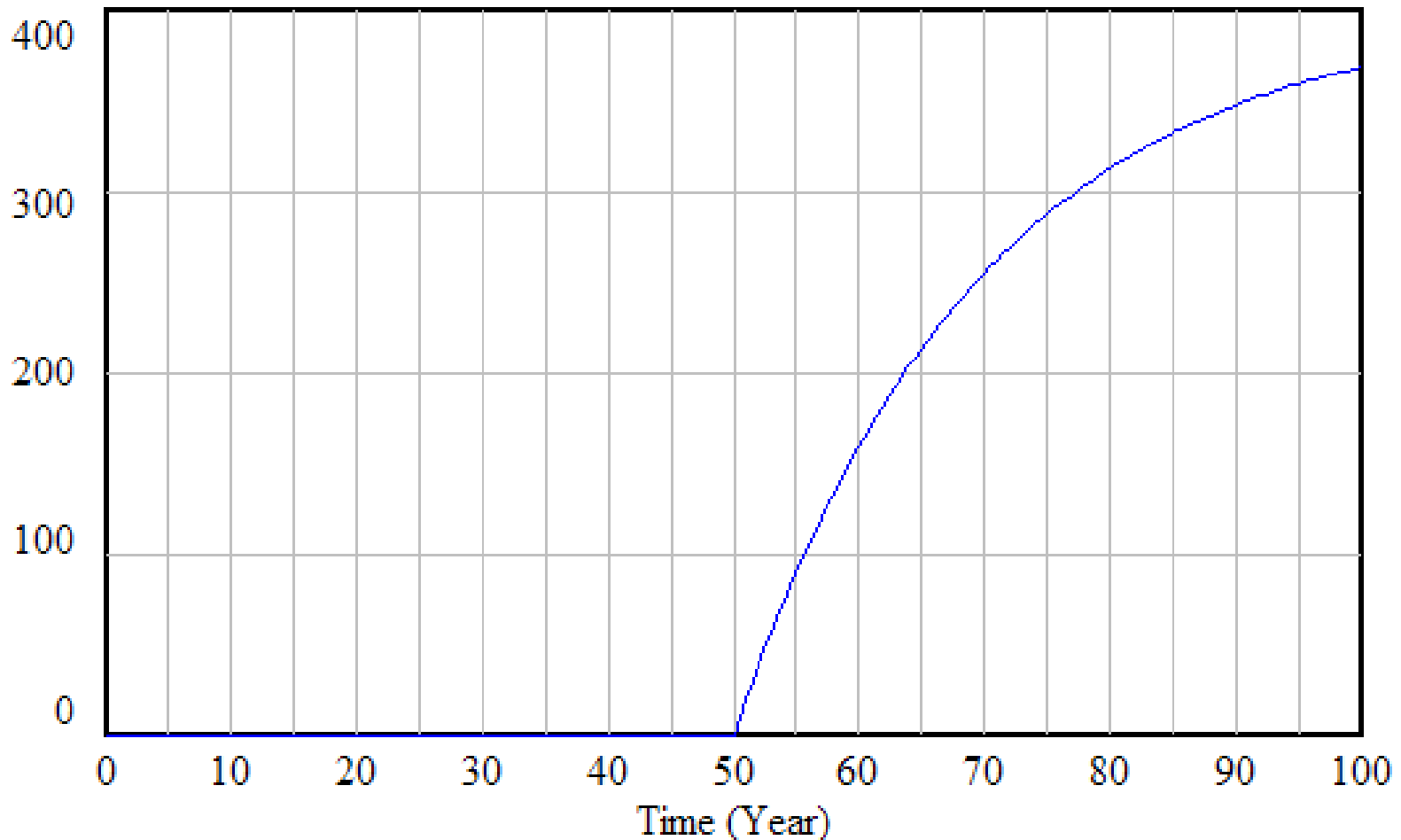
Immigration : Step Function 0 to 20 Initial Stock Empty  
Deaths : Step Function 0 to 20 Initial Stock Empty

How would this change with alpha?

# Stock Starting Empty?

## Value of *Stock* (Alpha=.05)

People (x)

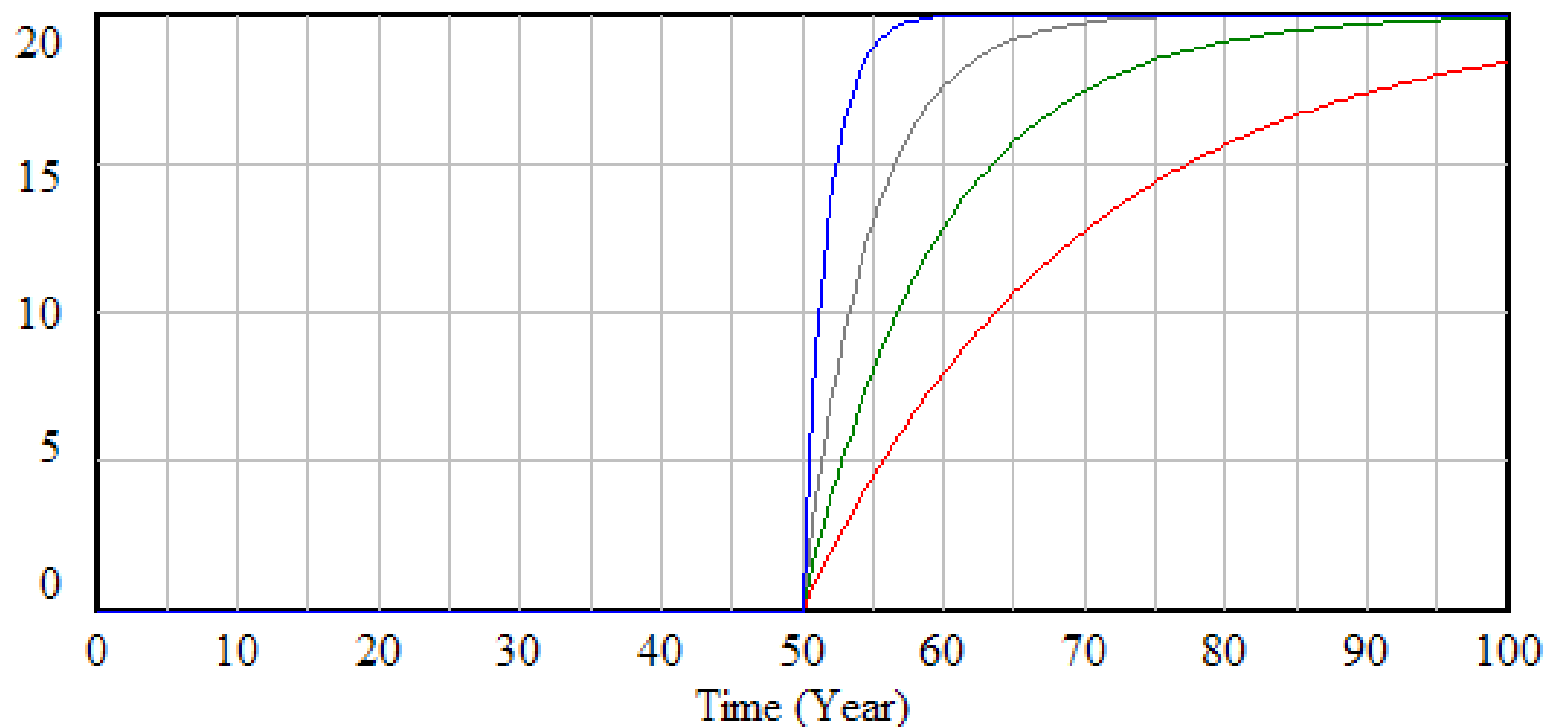


"People (x)" : Step Function 0 to 20 Initial Stock Empty

How would this change with alpha?

# For Different Values of $(1/\alpha)$ Alpha Flow Rates (Outflow Rises until = Inflow)

Deaths



Deaths : Step Functions 2 yr delay —————

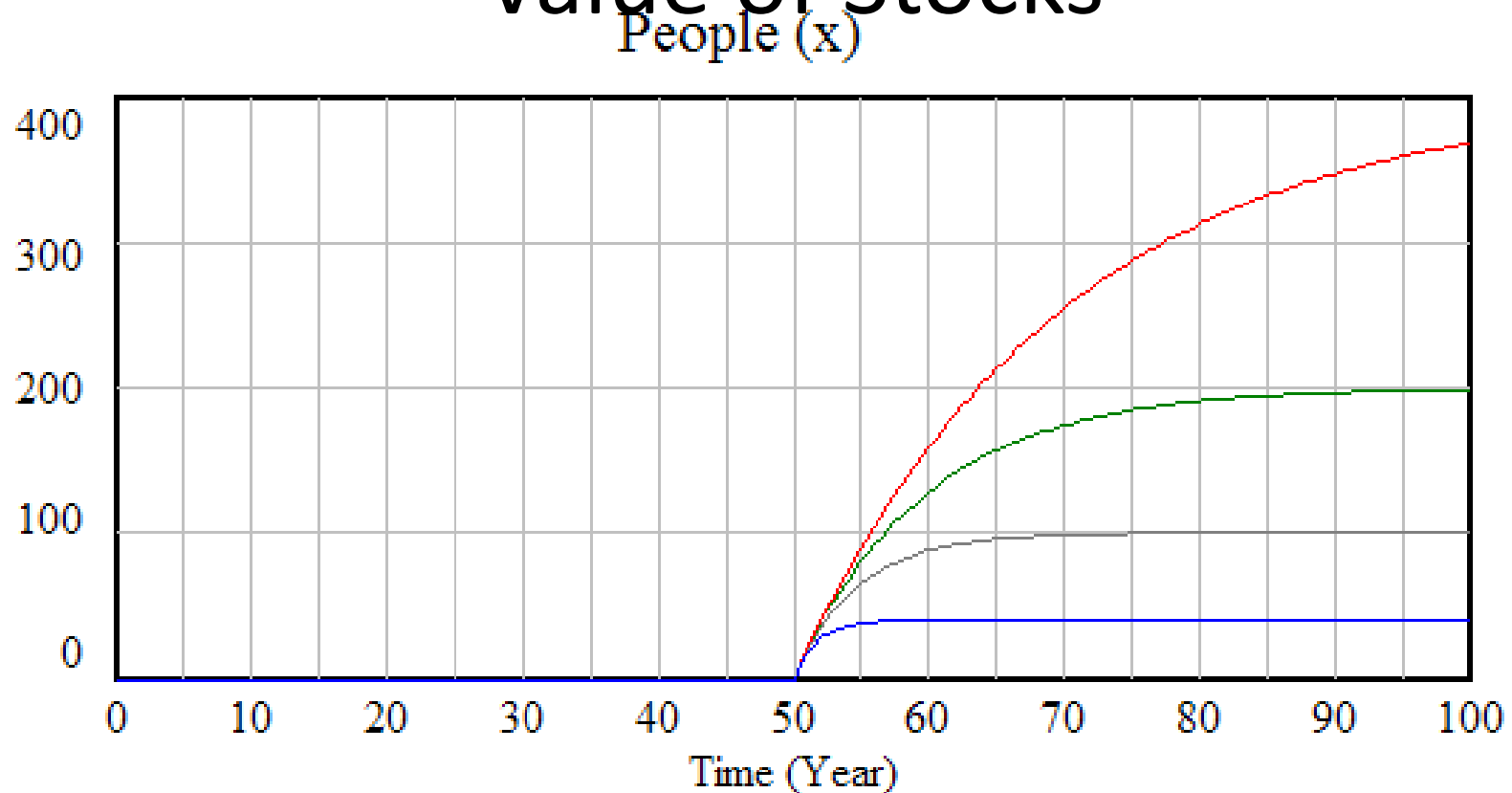
Deaths : Step Functions 20 yr delay —————

Deaths : Step Functions 10 yr delay —————

Deaths : Step Functions 5 yr delay —————

This is for the *flows*. What do stocks do?

# For Different Values of (1/) Alpha Value of Stocks



"People (x)" : Step Functions 2 yr delay

"People (x)" : Step Functions 20 yr delay

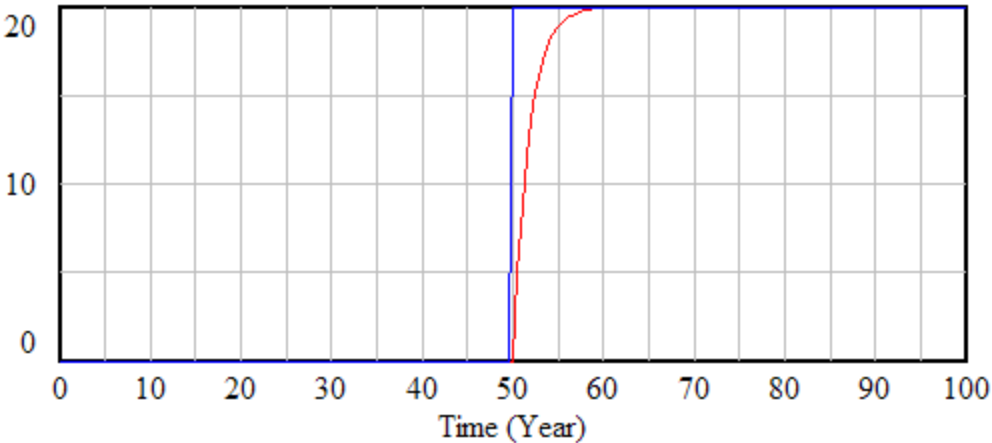
"People (x)" : Step Functions 10 yr delay

"People (x)" : Step Functions 5 yr delay

Why do we see this behaviour? A longer time delay (or smaller chance of leaving per unit time) requires  $x$  to be *larger* to make outflow=inflow

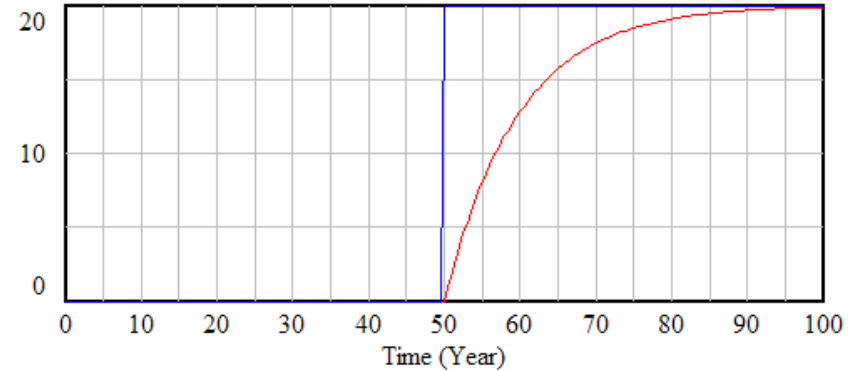
# Outflows as Delayed Version of Inputs

Inflow and Outflow



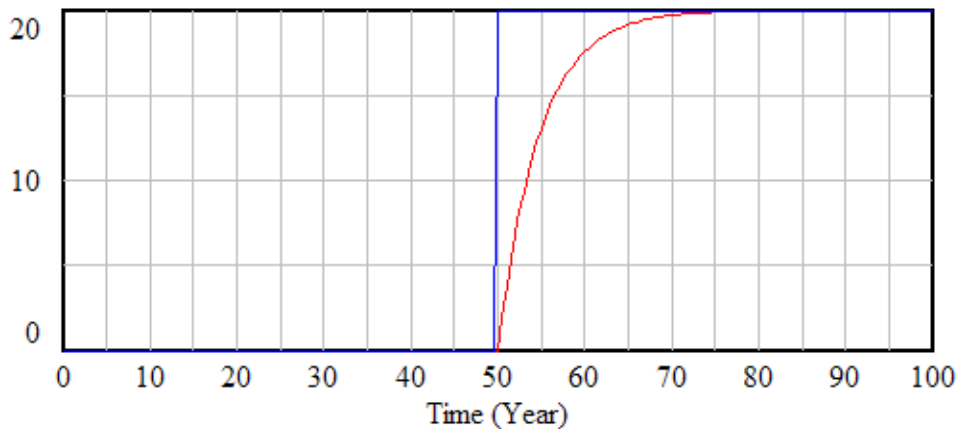
Immigration : Step Functions 2 yr delay —————  
Deaths : Step Functions 2 yr delay —————

Inflow and Outflow



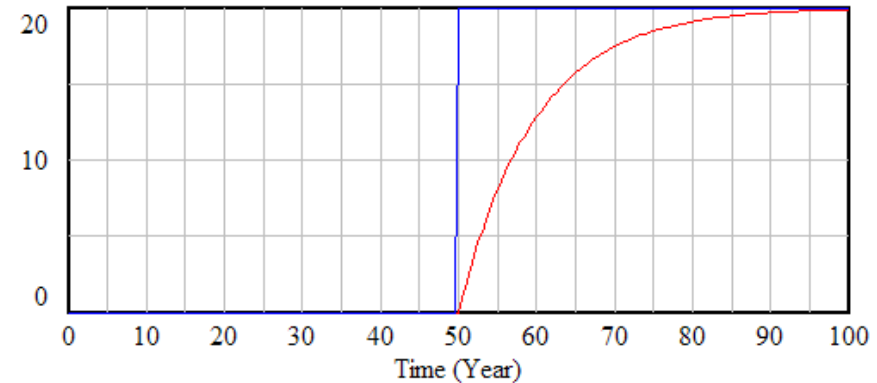
Immigration : Step Functions 10 yr delay —————  
Deaths : Step Functions 10 yr delay —————

Inflow and Outflow



Immigration : Step Functions 5 yr delay —————  
Deaths : Step Functions 5 yr delay —————

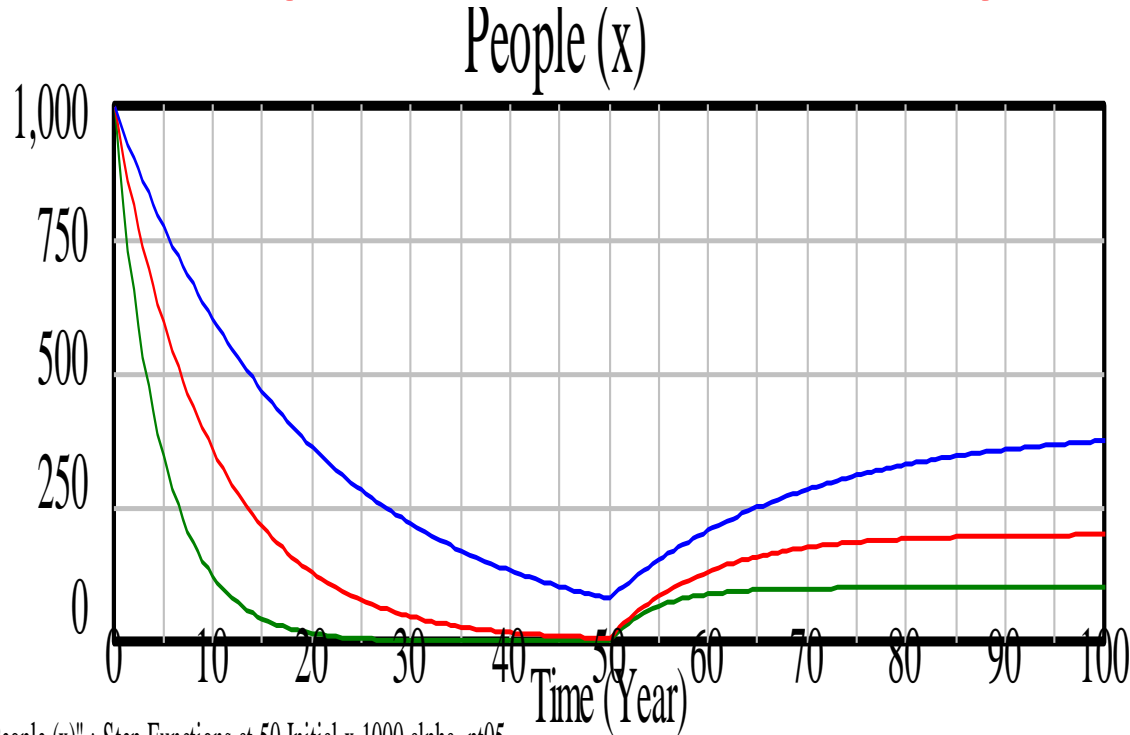
Inflow and Outflow



Immigration : Step Functions 10 yr delay —————  
Deaths : Step Functions 10 yr delay —————

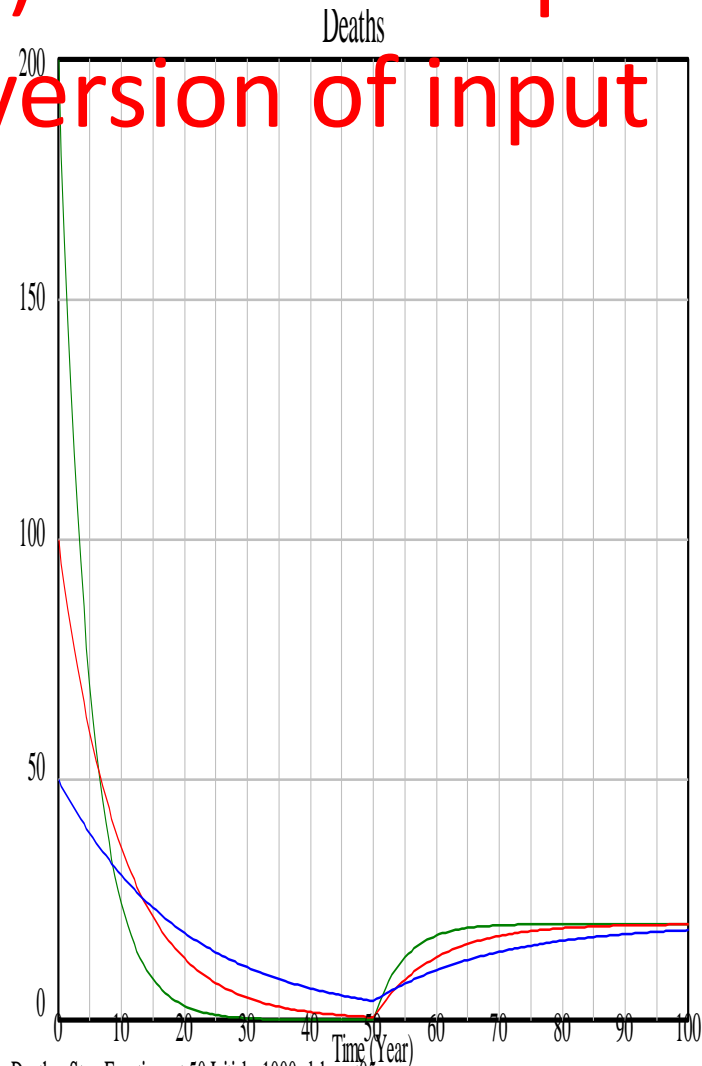
# What if stock doesn't start empty?

Decays at first (no inflow) & then output responds with delayed version of input



"People (x)": Step Functions at 50 Initial x 1000 alpha=0.05  
"People (x)": Step Functions at 50 Initial x 1000 alpha=0.1  
"People (x)": Step Functions at 50 Initial x 1000 alpha=0.2

— People (x) alpha=0.05  
— People (x) alpha=0.1  
— People (x) alpha=0.2

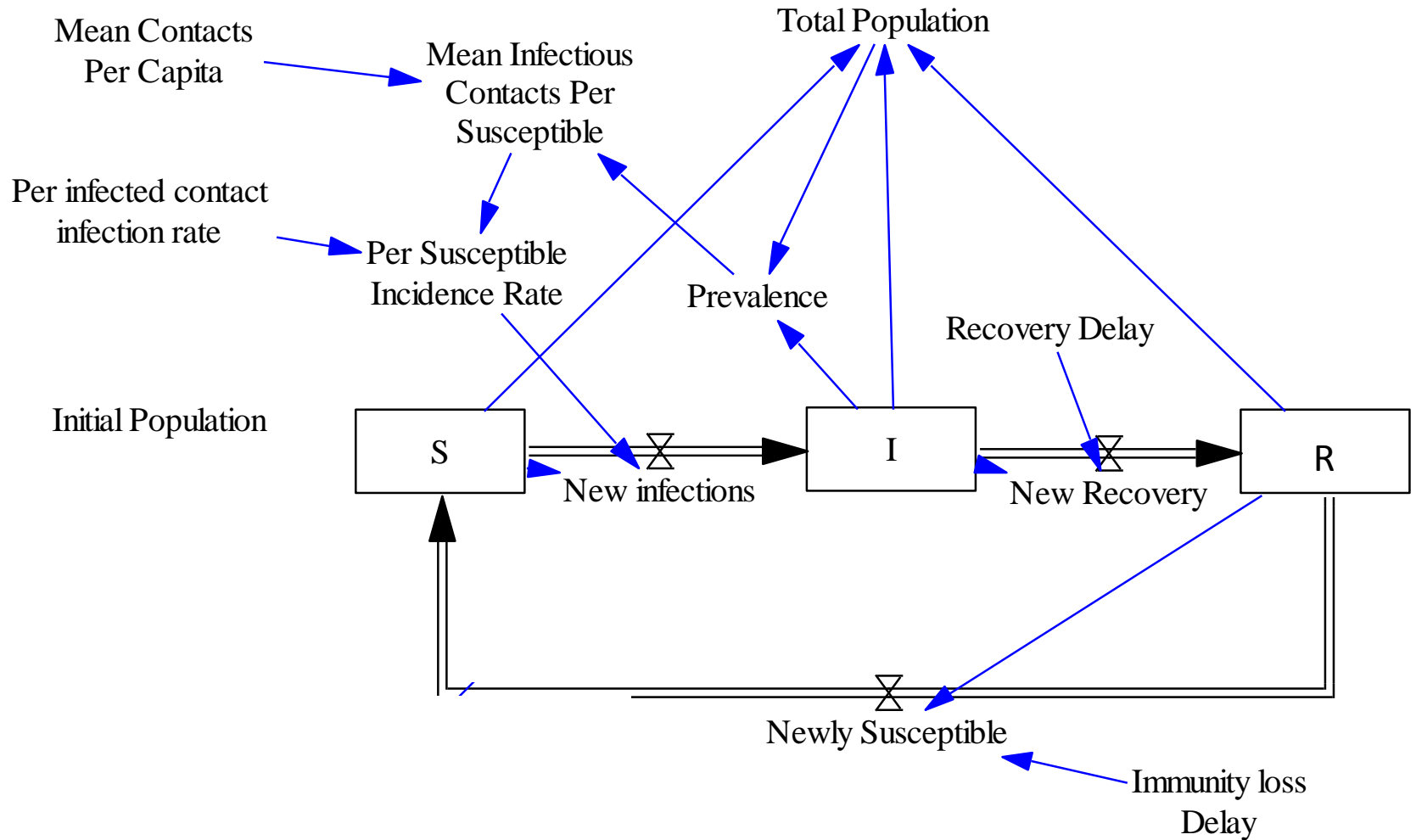


Deaths: Step Functions at 50 Initial x 1000 alpha=0.05  
Deaths: Step Functions at 50 Initial x 1000 alpha=0.1  
Deaths: Step Functions at 50 Initial x 1000 alpha=0.2

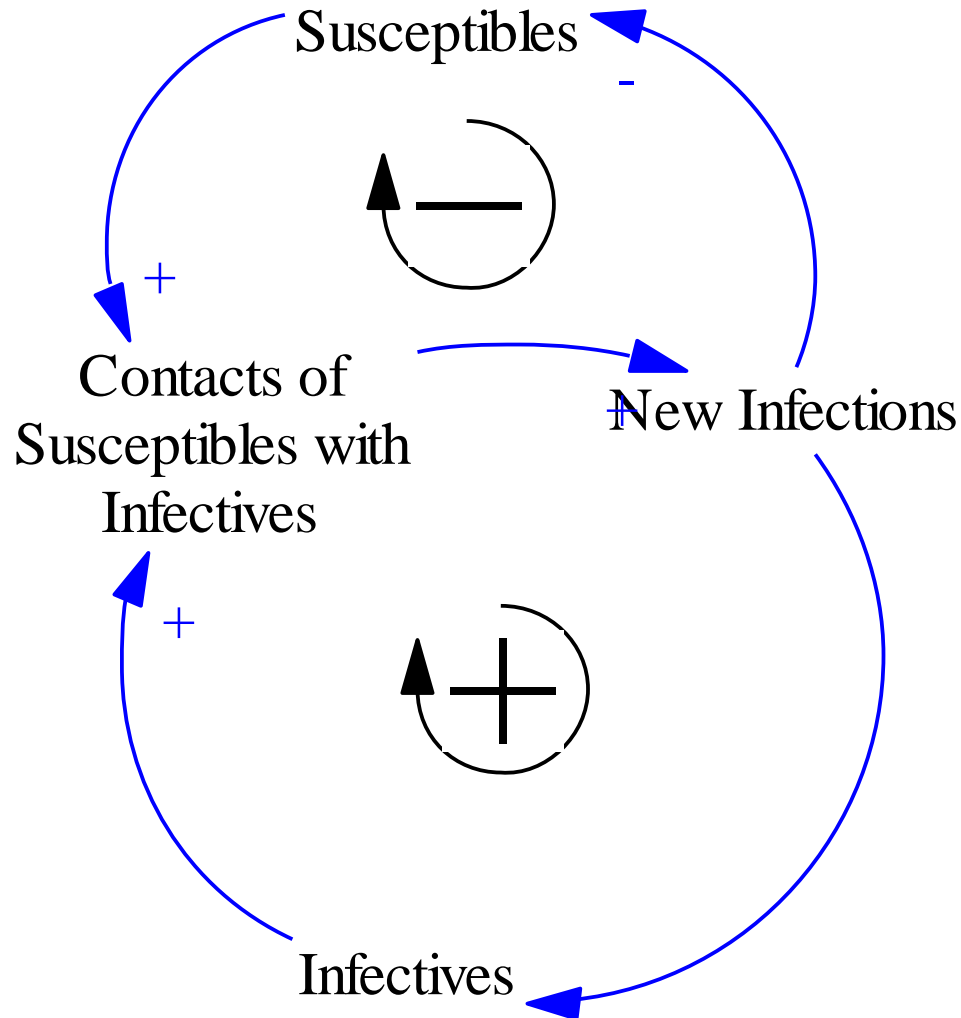
— Deaths alpha=0.05  
— Deaths alpha=0.1  
— Deaths alpha=0.2



# Simple SIT Model

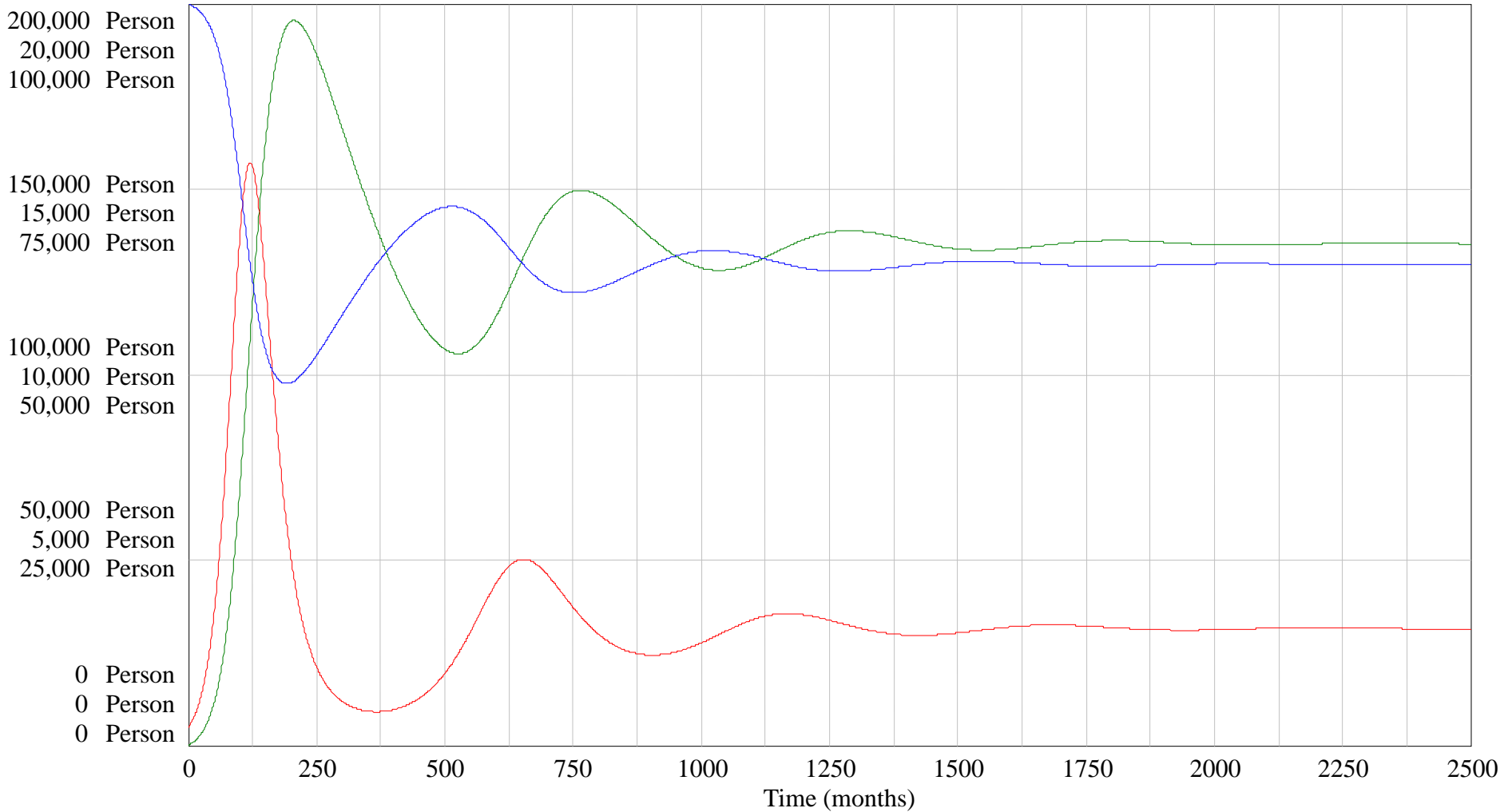


# Classic Feedbacks



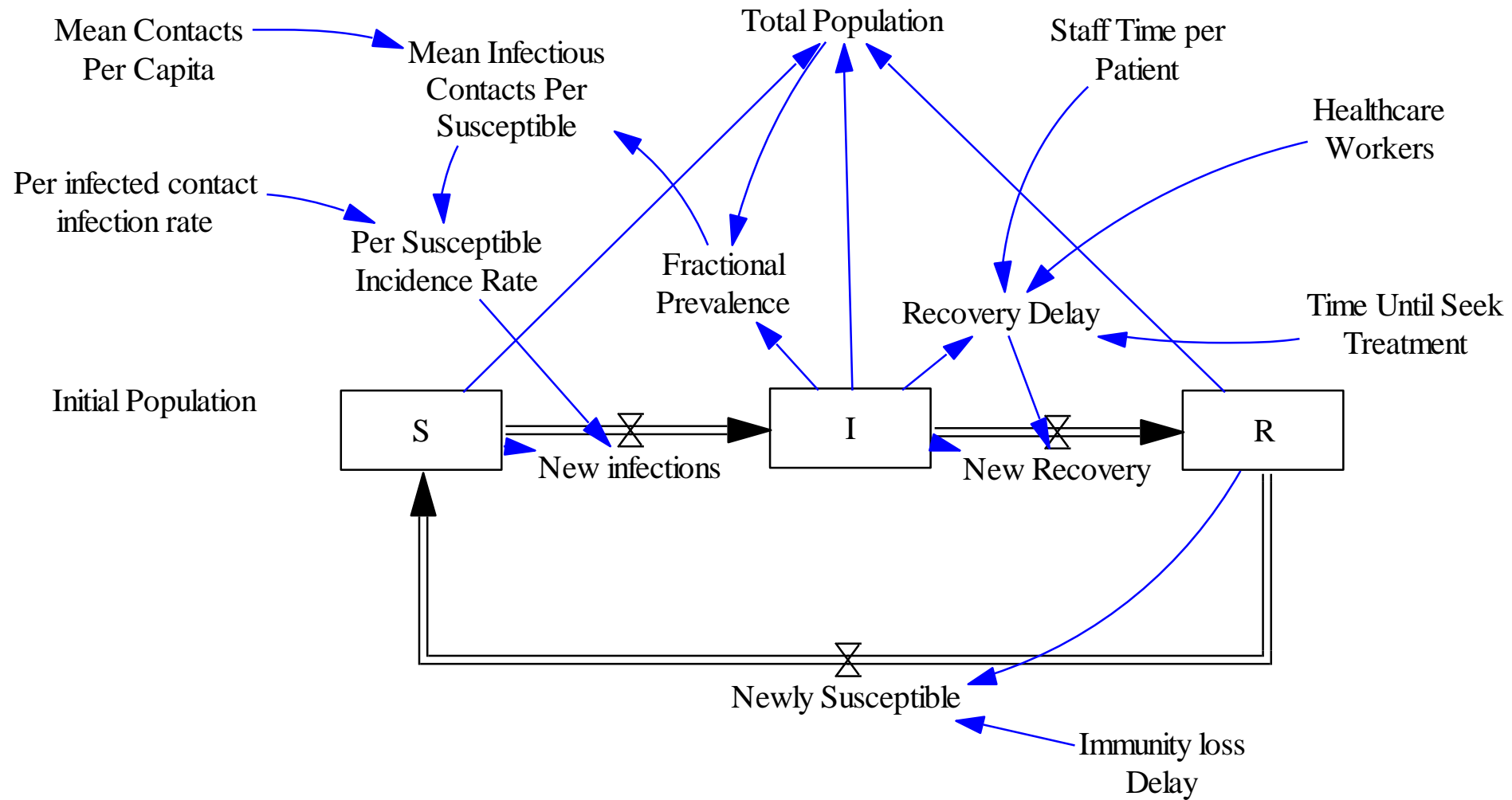
# Dynamics

State variables over time

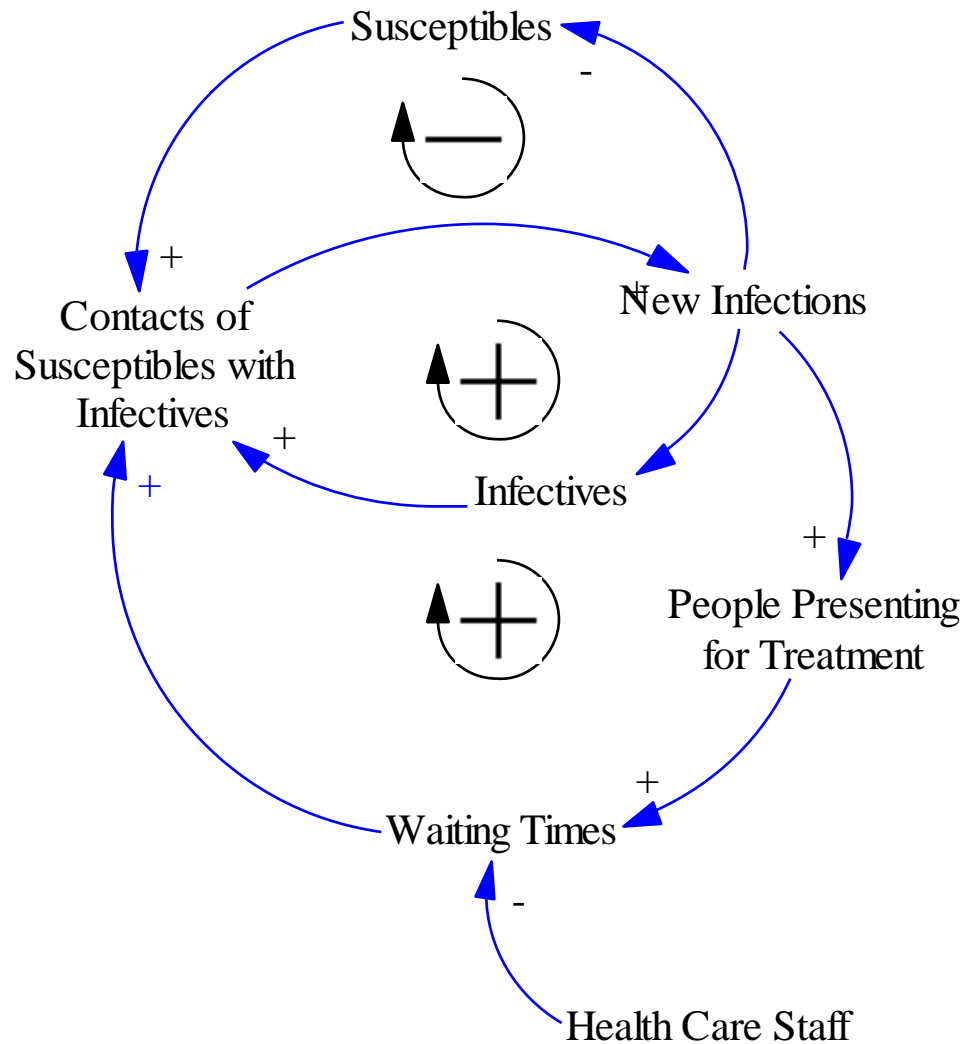


S : Alternative 30 HC Workers Exogenous Recovery Delay Person  
I : Alternative 30 HC Workers Exogenous Recovery Delay Person  
R : Alternative 30 HC Workers Exogenous Recovery Delay Person

# Broadening the Model Boundaries: Endogenous Recovery Delay

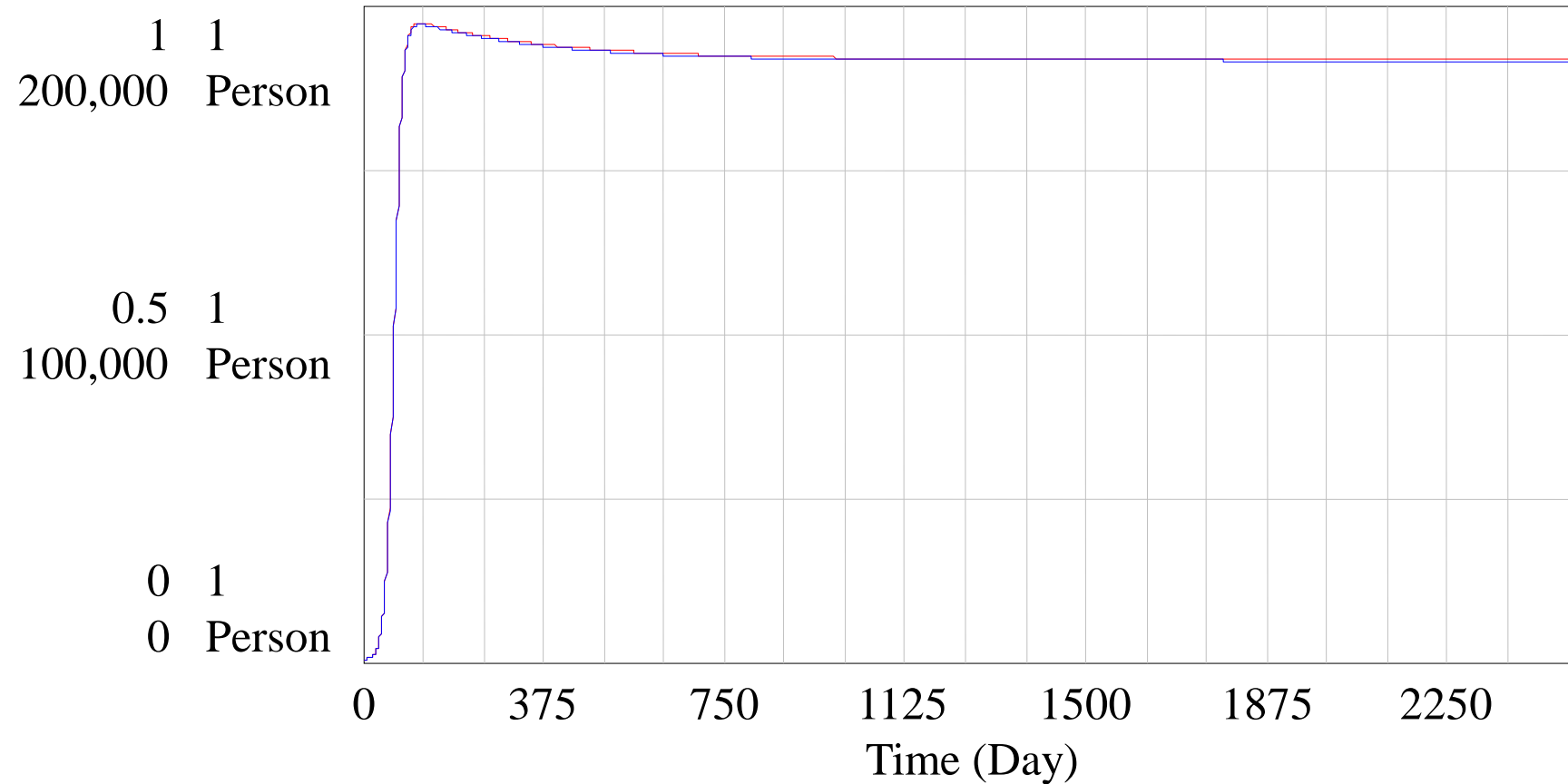


# Broadening the Model Boundaries: Endogenous Recovery Delay



# A Different Behaviour Mode

## Prevalence, Infectious



Prevalence : Baseline 30 HC Workers ————— 1  
I : Baseline 30 HC Workers ————— Person

# Structure as Shaping Behaviour

- System structure is defined by
  - Stocks
  - Flows
  - Connections between them
- Nonlinearity: The behaviour of the whole is more than the sum of the behaviour of the parts
  - “Emergent” behaviour would not be anticipated from simple behaviour of each piece in turn
- Stock and flow structure (including feedbacks) of a system determines the qualitative behaviour modes that the system can take on