Dimensional Reasoning & Dimensional Consistency Testing

Nathaniel Osgood

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Talk Outline

- Motivations & Background
- Dimensional Systems
- Dimensional Analysis
- Examples
- Discussion

Motivations

- General
  - Dimensional analysis (DA) critical historically for
    - Scoping models
    - Formulating models
    - Validating models
    - Calibrating models
  - Systems modeling community has made important but limited use of DA
  - Strong advantages from & opportunities for improved DA use
- Specific
  - Performance concerns for public health models

Big Ways Dimensional Analysis Can Help You Model

- Preventing common mistakes: Spotting models that are “nonsense” b/c dimensionally inconsistent
- Thinking through what and how parameters need to be used in a formula
- Reducing the number of parameters that are required for calibration & sensitivity analysis
- Creating models with smaller populations that can be used to understand the behavior of models with larger populations
- Explaining power law scaling relationships

Dimensions and Units

- *Dimensions* describe semantic category of referent
  - e.g. Length/Weight/Pressure/Acceleration/etc.
  - Describe referent
  - Independent of size (or existence of) measure
  - No conversions typical between dimensions
  - A given quantity has a unique dimension
- *Units* describe references used in performing a particular measurement
  - e.g. Time: μSeconds/Weeks/Centuries
  - This is *metadata*: Describes measured value
  - Relates to a *particular dimension*
  - Describe measurement of referent
  - Dimensional constants apply between units
  - A given quantity can be expressed using many units
  - Even dimensionless quantities can have units

Units & Dimensions

- Frequency
  - Dimension:1/Time
  - Units: 1/Year, 1/sec, etc.
- Angle
  - Dimension: “Dimensionless” (1, “Unit”)
  - Units: Radians, Degrees, etc.
- Distance
  - Dimension: Length
  - Units: Meters/Fathoms/Li/Parsecs

Dimensional Systems

- Synthetic: No canonical dimensional systems
  - Can often sharpen the dimensional analysis of a problem by clever selection of a dimensional system
- Dimensional systems can have as few as 1 and arbitrarily many dimensions
- Tradeoffs
  - Too many dimensions limits strength of dimensional analysis
  - Too few dimensions results in excessive overloading of identical dimensions for physically distinct

Structure of Dimensional Quantities

- Dimensional quantity can be thought of as a pair

  (value, m) where value $\in \mathfrak{R}$ and m $\in \mathfrak{R}^d$

- Quantity's dimension/units can be represented as
  - Products of powers of "reference" dimensions/units

    Suppose we have a model with Persons (P), Time (T), Money ($), Length (L)

    Incidence: Person/Time = $P^1T^{-1} = \$^0T^{-1}P^1L^0$

    Cash flow: $/Time = $\$^1T^{-1} = \$^1T^{-1}P^0L^0$

    Office space Cost: $/Area/Time : $\$^1T^{-1}P^0L^{-2}$

    Rate of water flow: Volume/Time: $\$^0T^{-1}P^0L^3$
  - Vectors in a *d* dimensional vector space (of ref. dimens.)
    - Each index in the vector represents the exponent for that reference dimension/unit

- Dimensional quantities have operations that are related to but more restricted than for e.g. $\mathfrak{R}$

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Dimensionality & Unit Choice

- Exponent for dimension dictates the numerical value scaling required by *unit conversion*
  - Consider x=1 \$/ft and y=1 $\$/ft^2$
    - Consider converting from feet to meters (note cancellation of units in denominator & numerator)
      - $x=1\ \$/ft^1 * (1ft/1m)^1 \approx 3.208\ \$/m$
      - $y= 1\ \$/ft^2 * (1ft/1m)^2 \approx 10.764\ \$/m^2$
- If the dimensions of a quantity do not depend on a given dimension (i.e. the dimensional power is 0), it is invariant to changing that dimension
  - e.g. a board's length (dimension LT^0) does not depend on the unit of time

A Particularly Interesting Dimensionality: “Unit” Dimension

- Recall: Dimensions associated with quantities can be expressed as “product of powers”
- We term quantities whose exponents are all 0 as being of “unit dimension”
- Another term widely used for this is “Dimensionless”
  - This is somewhat of a misnomer, in that these quantities do have a dimension – just a very special one
    - Analogy: calling something of length 0 “lengthless”

Dimensions & Scaling Under Change of Units

- A *dimensionless* quantity holds the same value regardless of measurement system
- ***A dimensionless quantity maintains the same numeric value regardless of measurement system***
  - e.g. Fraction = 0.1 (Unit Dimension)
    - 100 ft^2/1000 ft^2 =0.1
    - 0.92903 m^2/ 9.2903 m^2=0.1

    i.e. its value is invariant of our choice of length units
  - Likelihood (probability) (Unit Dimension)
- **Such quantities are independent of unit choice**

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Stock-Flow Dimensional Consistency

- Invariant: Consider a stock and its inflows and outflows. For any flow, we must have
  [Flow]=[Stock]/Time
- This follows because the Stock is the integral of the flow
  - Computing this integral involves summing up many timesteps in which the value being summed is the flow multiplied by time.

Seeking Hints as to the Dimension Associated w/a Quantity

- How is it computed in practice?
  - What steps does one go through to calculate this? Going through those steps with dimensions may yield a dimension for the quantity
- Would its value need to be changed if we were to change diff units (e.g. measure time in days vs. years)?
- Is there another value to which it is converted by some combination with other values?
  - If so, can leverage knowledge of dimensions of those other quantities

Computing with Dimensional Quantities

- To compute the dimension (units) associated with a quantity, perform same operations as on numeric quantities, but using dimensions (units)
- We are carrying out the same operations in parallel in the numerics and in the dimensions (units).
  - With each operation, we can perform it twice
    - Once on the numerical values
    - Once on the associated dimensions

Example

$$\frac{a+(b*c)}{d}$$

Suppose further that

[a]: Person

[b]: Person/Time

[c]: Time

[d]: $

To compute the dimensions, we proceed from "inside out", just as when computing value

- [b*c]=[b]*[c]=

(Person/Time)*Time=Person

- [a+(b*c)]=[a]+[b*c]=Person +Person=Person
- Thus, the entire expression has dimension

  [a+(b*c)/d] = [a+(b*c)/d]/[d] =Person/$

Dimensional Homogeneity

- There are certain computations that are dimensionally inconsistent are therefore meaningless
- Key principle: Adding together two quantities whose dimensions differ is dimensionally “inhomogeneous” (inconsistent) & meaningless
- By extension

  a^b is only meaningful if b is dimensionless

  Derivation: $a^b = \left(\frac{a}{e}e\right)^b = \left(\frac{a}{e}\right)^b e^b = \left(\frac{a}{e}\right)^b \left(1 + b + \frac{b^2}{2} + \frac{b^3}{3 \cdot 2 \cdot 1} + \cdots\right)$

  The expression on the right is only meaningful if [b]:1

Dimensional Homogeneity: Distinctions

- Adding items of different dimensions is semantically incoherent
  - 1 $ + 1 Person ?=
  - 1 m + 1 sec ?=
  - Fatally flawed reasoning
- Adding items of different units but the same dimension *is* semantically sensible but numerically incorrect
  - Requires a conversion factor

Adding Numbers of Different Units

- Consider 1 second + 1 minute
  - The answer to this is *not* simply 2 (i.e. 1+1) !

- To convert, we have (note cancellation)

1 second + 1 minute =

1 second + 1 minute * (60 seconds / minute)

= 1 second + 60 seconds = 61 seconds

Many Common Modeling Errors are Dimensional Errors

- Setting value of a flow equal to the value of a stock (or a fraction of a stock)
- Subtracting a flow from a stock
- Comparing a stock & flow
- For a flow, multiplying (rather than dividing) the value of a stock by a time constant
- For a flow, dividing (rather than multiplying) the value of a stock by hazard (likelihood per unit time)

Example: Classic SIR model

$$\dot{S} = -cS\left(\frac{I}{S+I+R}\right)\beta$$

$$\dot{I} = cS\left(\frac{I}{S+I+R}\right)\beta - \frac{I}{\mu}$$

$$\dot{R} = \frac{I}{\mu}$$

- Variables Dimensions

  [S]=[I]=[R]: Person

  [β]: 1 (A likelihood!)

  [c]: (Person/Time)/Person=1/Time

  (Just as could be calculated from data on contacts by *n* people over some time interval)

  [μ]: Time

Note that the force of infection $\lambda = c\left(\frac{I}{S+I+R}\right)\beta$ has units 1/Time, which makes sense

- Firstly, multiplying it by S must give rate of flow, which is Person/Time
- Secondly, the reciprocal of such a transition hazard is just a mean duration in the stock, which is a Time => dimension must be 1/Time

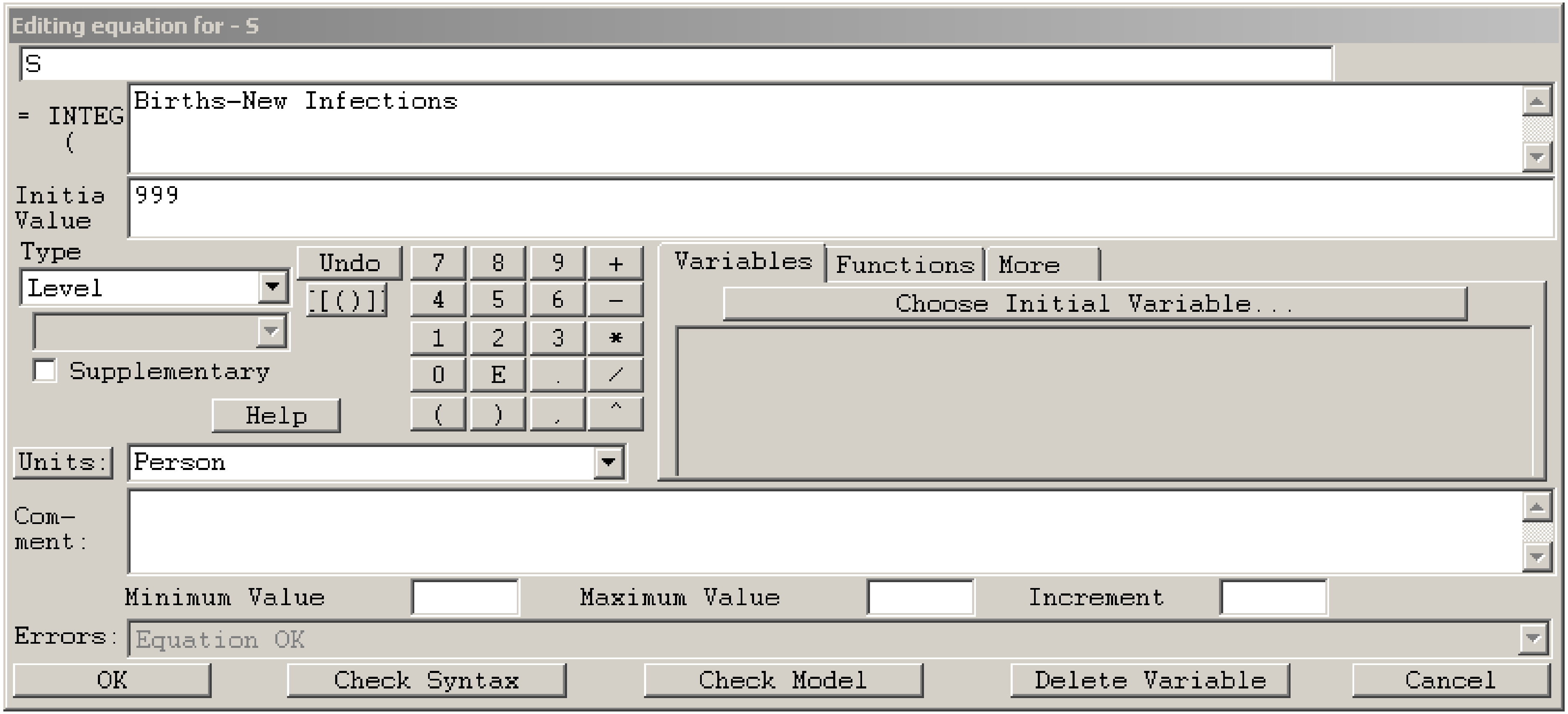
Vensim Interface

- Vensim will perform dimensional simplification via simple algebra on dimensional expressions
  - E.g. Person/Person is reduced to 1
- In some Vensim modes, when the mouse hovers over a variable, Vensim will show a pop-up “tab tip” that shows the dimension for that variable
- Vensim can check many aspects of dimensional consistency of a model

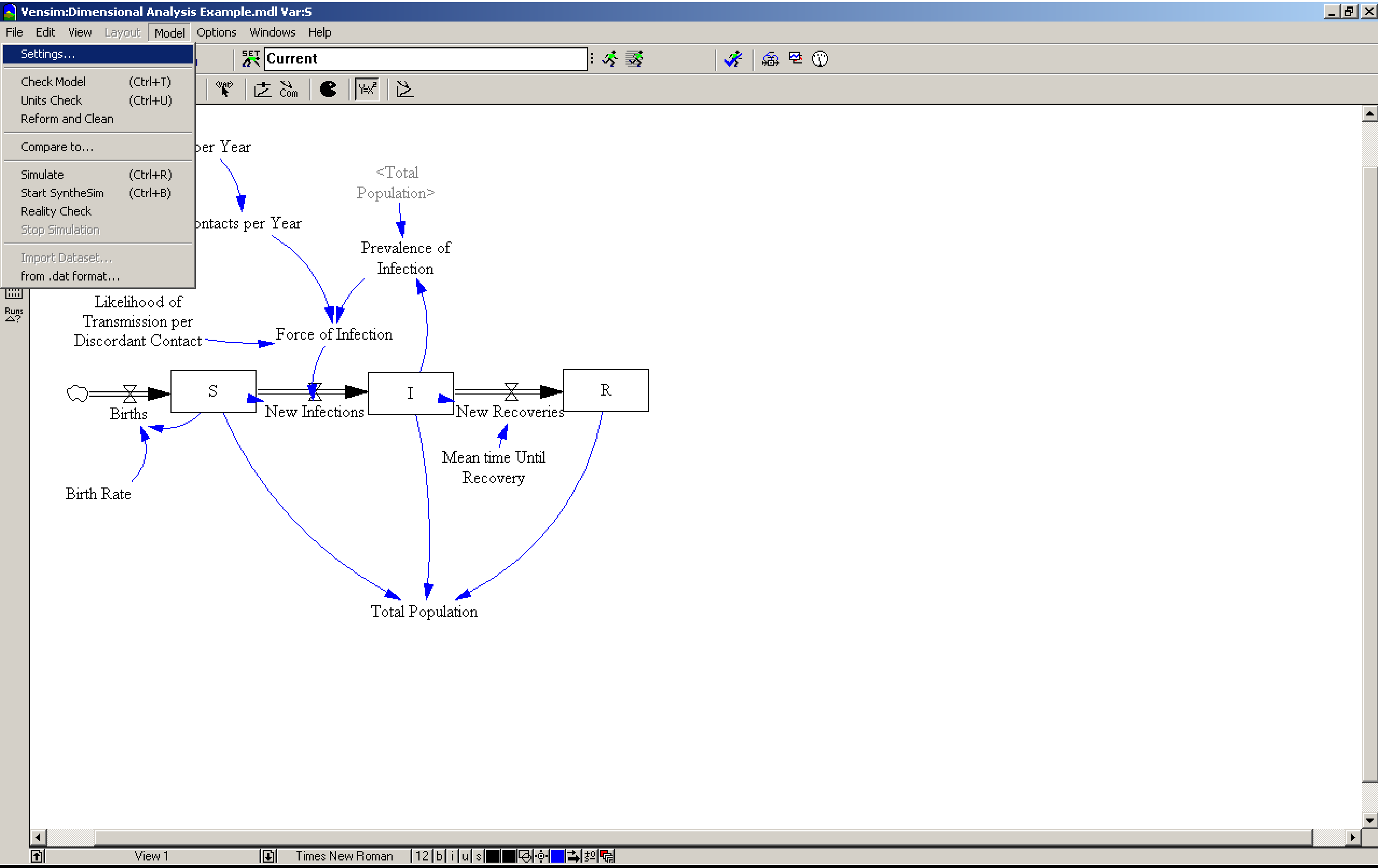
Vensim Capabilities

- Associate variables with units
- Define new units (beyond built-in units)
  e.g. Person, Deer, Bird, Capsule
- Define unit equivalence
  e.g. “Day”, “Days”

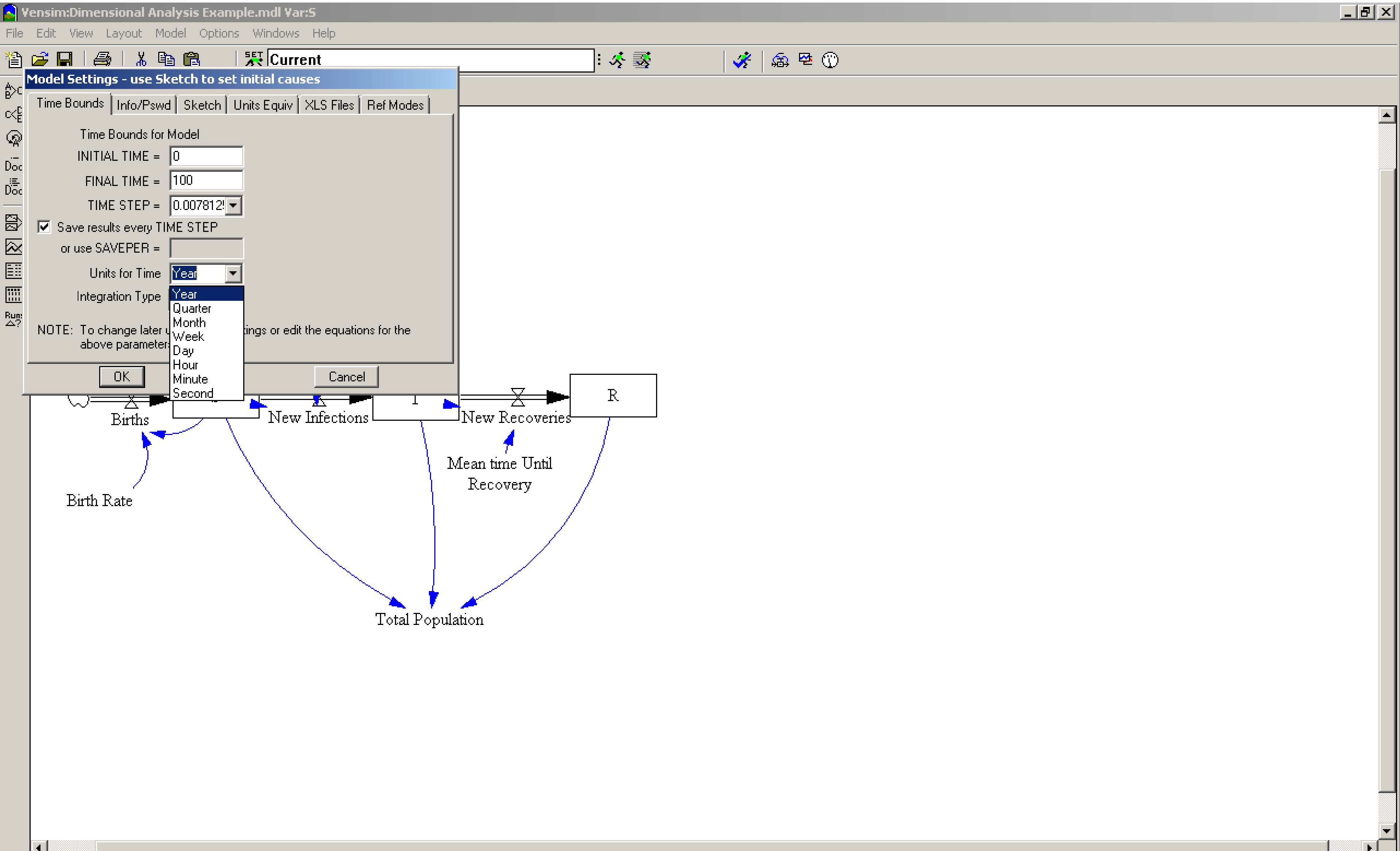
Indicating Units Associated with a Variable in Vensim



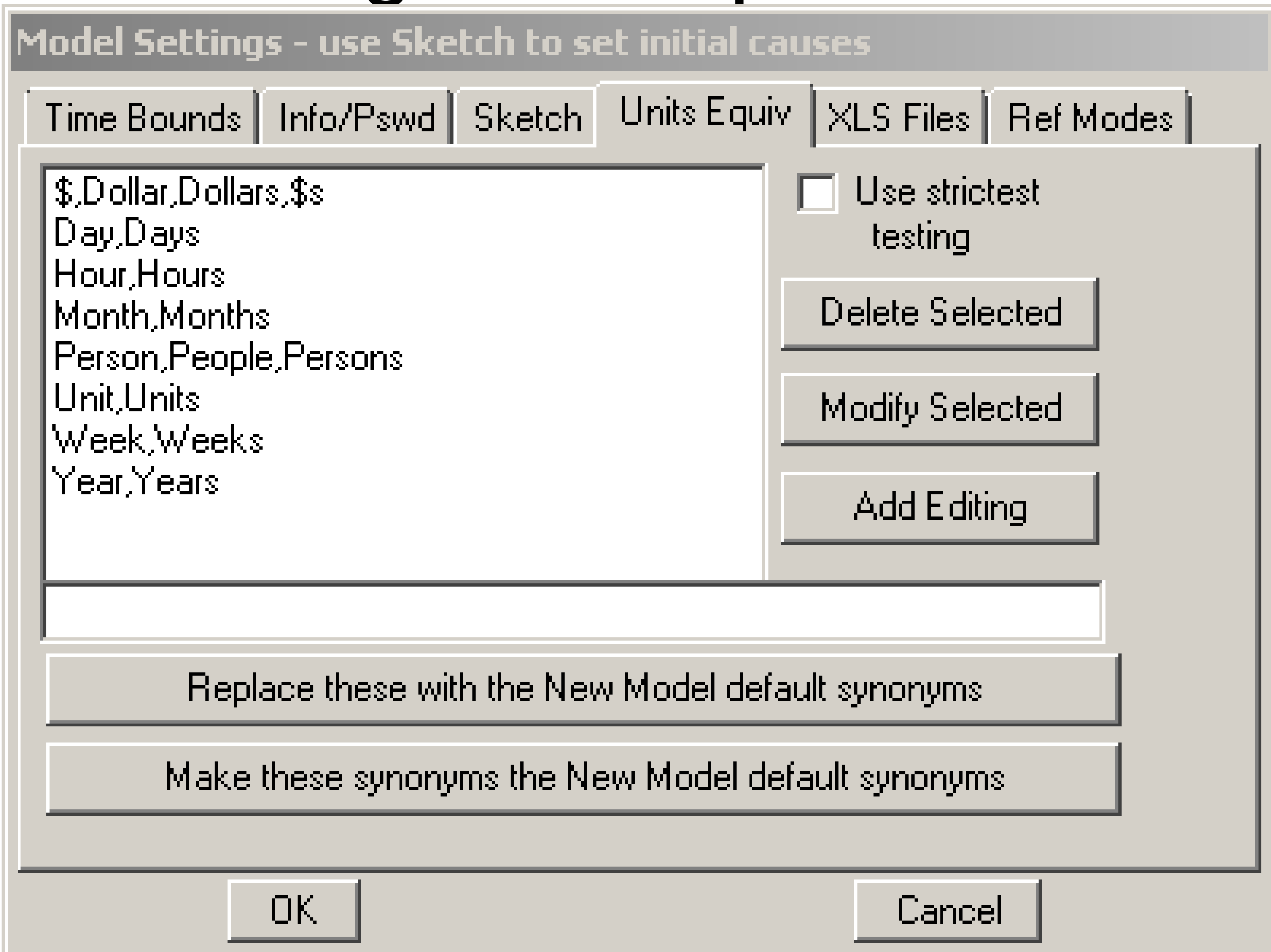
Accessing Model Settings



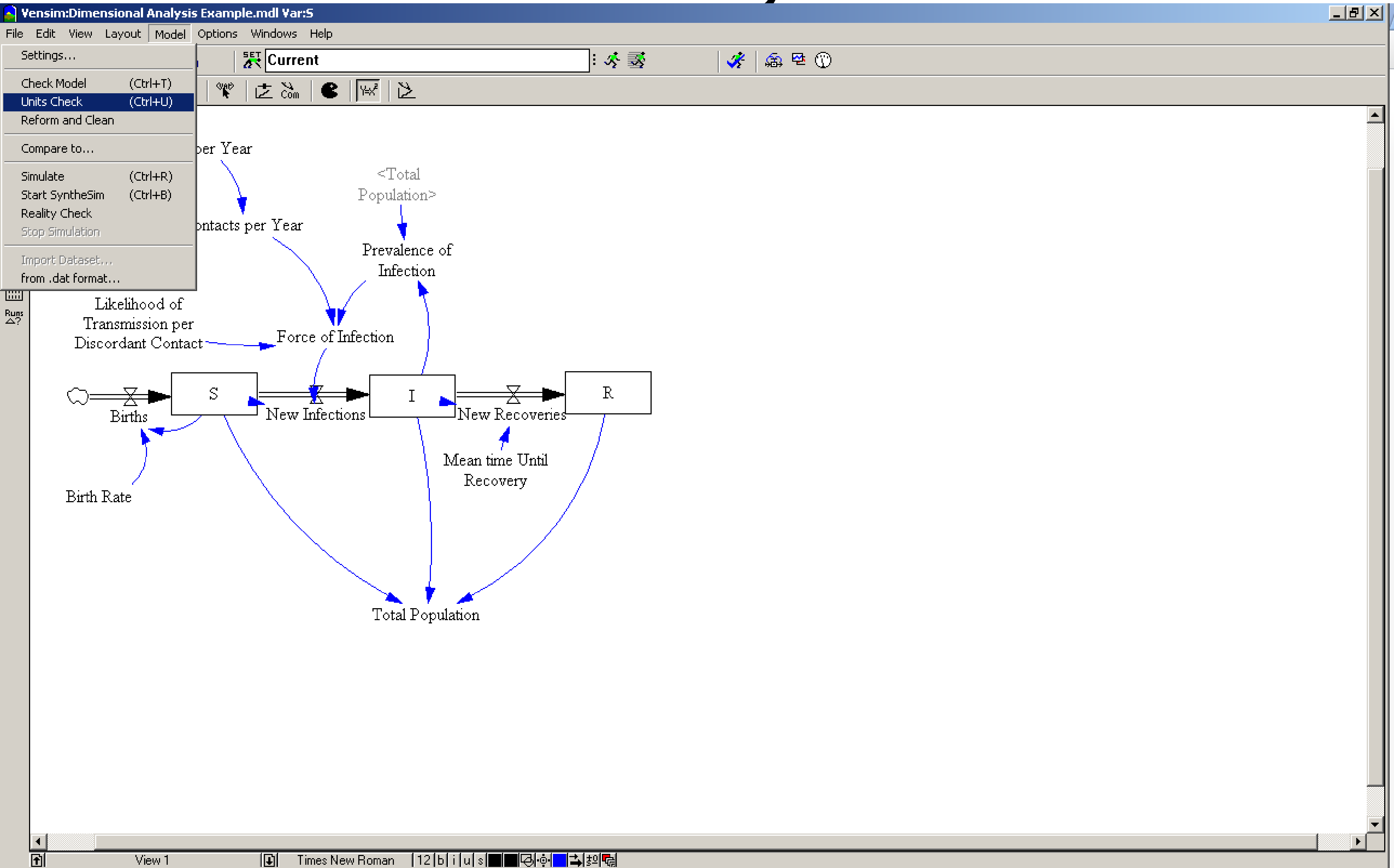
Choosing Model Time Units



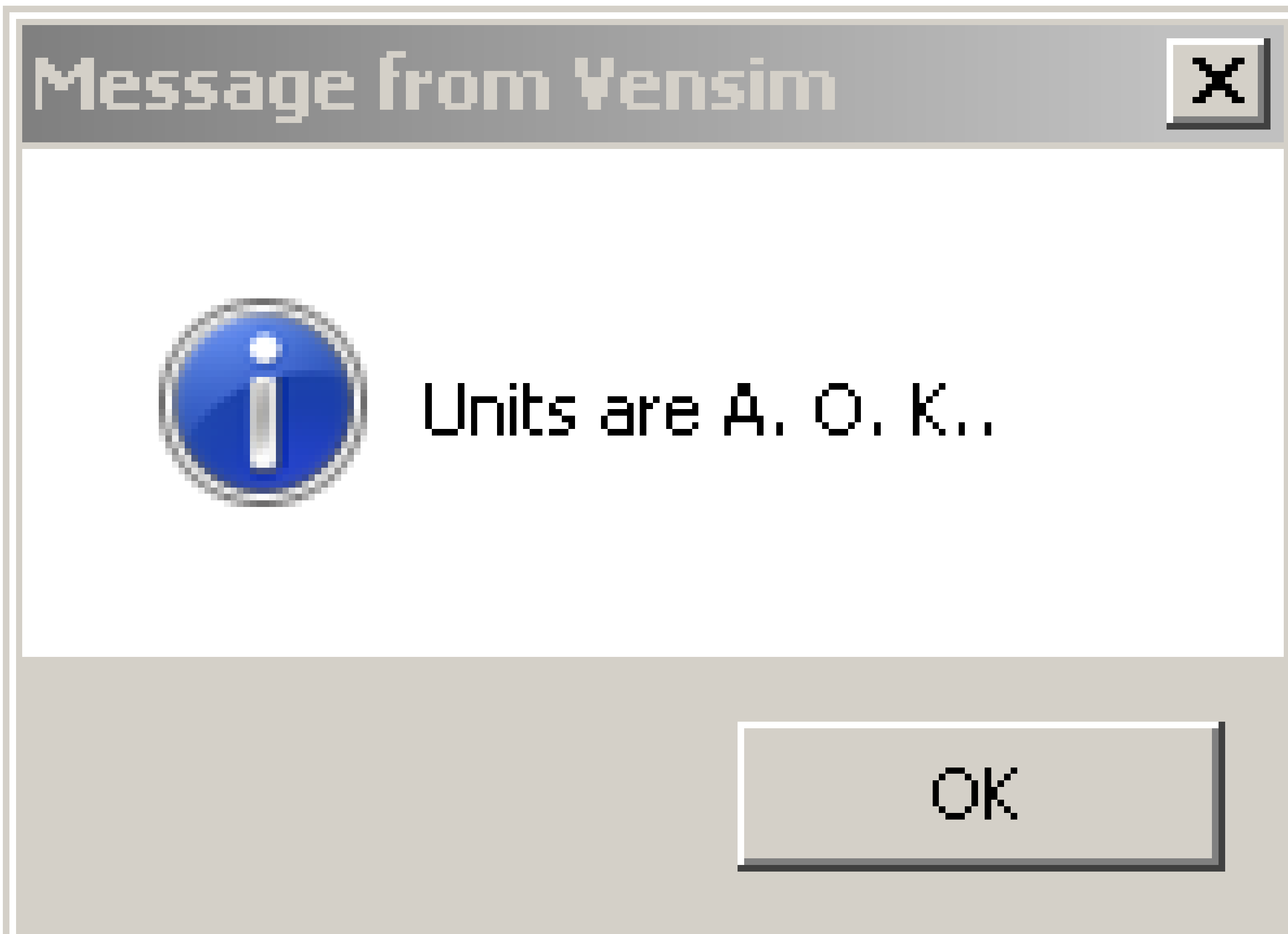
Setting Unit Equivalence



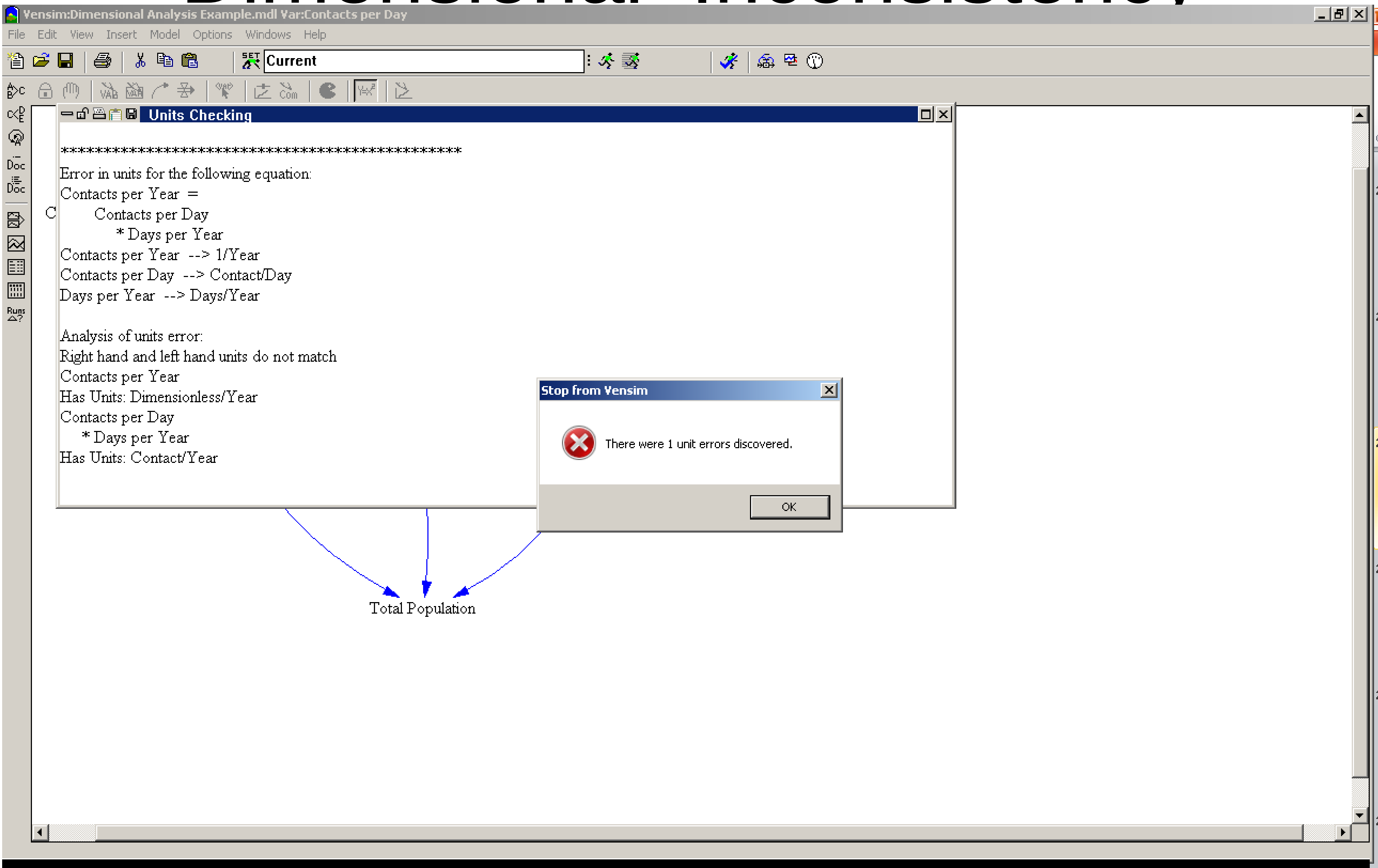
Requesting a Dimensional Consistency Check



Confirmation of Unit Consistency



Indication of (Likely) Dimensional Inconsistency



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- Preventing common mistakes: Spotting models that are “nonsense” b/c dimensionally inconsistent
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Dimensional Notation

- Within this presentation, we'll use the notation
  - [x]: D to indicate quantity x is associated with dimension D
- For example,
  - [x]: $
  - [y]: Person/Time
  - [z]: 1

Lotka Volterra model

$$\dot{H} = -\beta HF + \alpha H$$

$$\dot{F} = \gamma HF - \delta F$$

- Variables Dimensions
  [β]: 1/(Fox * Time)
  [γ]: 1/(Hare * Time)
  [δ],[α]: 1/Time
- Cf: Frequency of oscillations: [λ] : (1/Time)
  - Clearly cannot depend on β or γ, because
    - These parameters would introduce other dimensions
    - Those dimensions could not be cancelled by any other var.
- The exponent of Time in [λ] is -1
- By symmetry, the period must depend on both α and δ, which suggests

$$\sqrt{\alpha\delta}$$

Pendulum

- Variables
  - Length [d]: Length
  - Mass [m]: Mass
  - Gravitational Acceleration [g]: Length/$Time^2$
- Dependent variable: Period of oscillation t:[Time]
- Since dimensions of no other quantity can cancel dimension [Mass] of m, the formula for t must not depend on m
- Formula for t can only depend on d & g
- To give dimension $[Time]^1$ must be proportional to

$$\sqrt{\frac{d}{g}}$$

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Buckingham Π Theorem

- Any dimensionally homogenous equation involving n variables in k dimensions can be re-expressed as an constituent equation of fewer ($\geq$ n-k) dimensionless variables
  - Key: Exploit dimensional homogeneity restriction
- Similitude: ‘dimensionally similar’ systems share
  - Same constituent equation (possibly vector)
  - Identical values for each dimensionless variable

Buckingham Π Theorem: Benefits

- Provides a way of reducing the complexity of the model formulation (n-k rather than n variables)
  - E.g. allows easier
    - State space (e.g. phase plane) analysis
    - Calibration analysis (fewer parameters to calibrate)
    - Sensitivity analysis (fewer parameters to vary)
- Provides a way of producing models that are independent of measurement systems
- Provides the key insight needed to build (reduced) scale models

Sensitivity Analysis & Calibration

- As long as we preserve the values of the dimensionless variables, the behavior of the model should be identical
- We can thus vary the (fewer) dimensionless variables rather than the greater number of original parameters

Recall: Classic SIR model

$$\dot{S} = -cS\left(\frac{I}{S+I+R}\right)\beta$$

$$\dot{I} = cS\left(\frac{I}{S+I+R}\right)\beta - \frac{I}{\mu}$$

$$\dot{R} = \frac{I}{\mu}$$

- Variables Dimensions

  [S]=[I]=[N]: Person (R = N-S-I)

  [β]: 1 (A likelihood!)

  [*c*]: (Person/Time)/Person=1/Time

  (Just as could be calculated from data on contacts by *n* people over some time interval)

  [μ]: Time

Parameters (6): N_0, S_0,I_0, β, *c*, *μ*

Note that the force of infection $\lambda = c\left(\frac{I}{S+I+R}\right)\beta = c\left(\frac{I}{N}\right)\beta$ has units 1/Time, which makes sense

- Firstly, multiplying it by S must give rate of flow, which is Person/Time
- Secondly, the reciprocal of such a transition hazard is just a mean duration in the stock, which is a Time => dimension must be 1/Time

Dimensional Matrix

	μ	I_0	S_0	β	N	c
Person	0	1	1	0	1	1
Time	1	0	0	0	0	-1
π_1	1	0	0	0	-1	1
π_2	0	1	0	0	-1	0
π_3	0	0	1	0	-1	0
π_4	0	0	0	1	0	0

“Dedimensionalized” SIR model

$$\dot{S} = -cS\left(\frac{I}{S+I+R}\right)\beta$$

$$\dot{I} = cS\left(\frac{I}{S+I+R}\right)\beta - \frac{I}{\mu}$$

$$\dot{R} = \frac{I}{\mu}$$

- Variables Dimensions

  [S]=[I]=[R]: Person

  [β]: 1 (A likelihood!)

  [c]: (Person/Time)/Person=1/Time

  (Just as could be calculated from data on contacts by *n* people over some time interval)

  [μ]: Time

Parameters (4): $\frac{\mu c}{N}, \beta, \frac{I_0}{N}, \frac{R_0}{N}$

Note that the force of infection $\lambda = c\left(\frac{I}{S+I+R}\right)\beta$ has units 1/Time, which makes sense

- Firstly, multiplying it by S must give rate of flow, which is Person/Time
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Motivation: Performance Concerns for Individual-Based Modeling

- IBM modelers are increasingly asked to handle megapopulations (>10^6)
- Individual-based disaggregation: Population size heavily impacts runtime [$\Omega(n)$]
  - Unlike attribute based disaggregation [$\theta(n)$]
  - Dense networks can rise as $\theta(n)$
- Stochastics pronounced at individual level Þ Monte Carlo analyses often sought
- Slow runtime have high opportunity costs (inhibit insight, model refinement)
- *By building reduced-scale IBM, we can avoid the need for prohibitive runtimes*

(Reduced-)Scale Models

- Given that a given output depends on many factors, how to scale different parameters to create a comparable but smaller model?
- Naïvely, it may appear easy to create a reduced scale model by using a smaller population
- Non-obvious questions can arise
  - Should we reduce the size of the space in which the agents are embedded? If so, by how much?
  - If agents connections are distance-based, should we change the connection threshold?
  - Should the birth rate also be scaled down?
  - What if there are more than 1 population? How do they get scaled consistently?
  - How would timing of emergent behavior translate?

(Reduced-)Scale Models

- Given that a given output depends on many factors, how to scale different parameters to create a comparable but smaller model?
- Naïve reduced-scale models can be misleading
  - Non-linear models: Reducing the population from $p_{fs} \rightarrow p_{rs}$ does not reduce all outputs proportionately
- Principled answer: *Use dimensional analysis to derive model scaling laws*
  - Reduced scale models exploit Buckingham's Π Theorem

Properties of Scale Model

- A scale model is specific to the *dimension* of a dependent variable
- Can use same scale model for different quantities sharing dimensions e.g. for (1/Time)
  - Largest eigenvector
  - Oscillation frequency
  - Time density of zero-crossings
- Can change *units* of model freely – value of dimensionless products remain the same

Scale Model Construction

- Identify dimensions of all model quantities
- Identify dimension of dependent quantity
- Derive the dimensionless quantities
  - Parameterize matrices A, B and D in the dimensional set
  - Derive matrix C using $C = -\left(A^{-1}B\right)^T$
  - This provides us with the *model laws* for the system

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Deriving Reduced-Scale Model Parameters

- Derive Reduced-Scale Model Parameters
  - Set the expressions for the dimensionless quantities = for full- & reduce-size models
  - Set the values of free variables parameters of the reduced-size model as desired for adequate performance
  - Derive the values for the other reduced-size model parameters from the parameter values of the full-sized model & previously set reduced-size parameters
    - Left with equations relating dependent variables
- Measure the value of the dependent variable for the reduced-size model
- Deduce value that must hold for full-sized model

Simple Individual-based SIR Scale Model Construction

- Parameters
  - [β]: Transmission Rate: 1/(Person*Time)
  - [α] : Fractional Recovery Rate: 1/Time
  - [X_0] : Person
- Suppose we want to know the growth rate of infection [λ] : 1/Time at the beginning of an infection for a model with a large population (X_0), but cannot afford to run it
- We can build a scale model to find this

Determine Identify A,B,D blocks of Dimensional Matrix

	λ	β	α	X_0
Person	*0*	*-1*	*0*	*1*
Time	*-1*	*-1*	*-1*	*0*
π_1	*1*	*0*	*-1*	*0*
π_2	*0*	*1*	*-1*	*1*

- A & B just filled in per dimensions of parameters
- D is always identity

Completing the Dimensional Matrix

- We now need to complete the definition of the dimensionless variables
- C Block is determined as

$$-\left(A^{-1}B\right)^{T}=\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^{-1}\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}\right]^{T}=-\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix}\right]^{T}=-\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}^{T}=\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$$

- By inspection: “What combination of parameters are needed to cancel each of the parameters represented in the D block?”.
- Result:

	λ	β	α	X_0
Person	*0*	*-1*	*0*	*1*
Time	*-1*	*-1*	*-1*	*0*
π_1	*1*	*0*	*-1*	*0*
π_2	*0*	*1*	*-1*	*1*

Creating the Model Laws

	λ	β	α	X_0
Person	*0*	*-1*	*0*	*1*
Time	*-1*	*-1*	*-1*	*0*
π_1	*1*	*0*	*-1*	*0*
π_2	*0*	*1*	*-1*	*1*

- From the above, the dimensionless variables are

$$\frac{\lambda}{\alpha} \quad \text{and} \quad \frac{\beta X_0}{\alpha}$$

- These variables must hold identical values for the full-sized and reduced-size models

$$\frac{\lambda_f}{\alpha_f} = \frac{\lambda_r}{\alpha_r} \qquad \frac{\beta_f X_{0_f}}{\alpha_f} = \frac{\beta_r X_{0_r}}{\alpha_r}$$

Solving the Scaling Relations

- We know all of the parameters of the full-sized problem – except for the dependent parameter (λ_f)
- But $\exists$ many scale models that would maintain the same value for π_1 and π_2
  - We need to choose |V|-|D|=4-2=2 parameters, picked for convenience (including performance)
  - We pick X_{0r} and β_r. Thus, we have

$$\frac{\beta_f X_{0_f}}{\alpha_f} = \frac{\beta_r X_{0_r}}{\alpha_r} \Rightarrow \alpha_r = \alpha_f \frac{\beta_r X_{0_r}}{\beta_f X_{0_f}} \quad \text{and} \quad \frac{\lambda_f}{\alpha_f} = \frac{\lambda_r}{\alpha_r} \Rightarrow \lambda_f = \frac{\lambda_r \alpha_f}{\alpha_r}$$

Completing the Experiment

- All parameters of the reduced-size model are now determined.
- We can now run the experiment & make measurements of λ_r.
- Finally, we can compute the value of λ_f from the measured value for λ_r.
  - This provides the information regarding the behavior of the full-sized model without the need to run that model

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Identify dimensions

- Model quantities
  - Death rates [μ] : 1/Days
  - Transmission rates [β_i] : 1/(Organism * Days)
    - β_h: [1/(Host * Days)]
    - β_v : [1/(Vector * Days)]
  - Migration rates: Π_i: [Organisms/Day]
    - Π_h: [Host / Day]
    - Π_v : [Vector / Day]
  - Mean recovery rates by infected hosts α : [1/Day]
- Dependent quantity of dimension 1/Time e.g.
  - Largest eigenvector [λ] encountered during first 100 days of epidemic

Determine Identify A,B,D blocks of Dimensional Matrix

	λ	Π_h	Π_v	μ_h	μ_v	α	β_h	β_v	S_{h0}	S_{v0}
Host	0	1	0	0	0	0	0	-1	1	0
Vector	0	0	1	0	0	0	-1	0	0	1
Time	-1	-1	-1	-1	-1	-1	-1	-1	0	0
π_1	1	0	0	0	0	0	0	1	1	0
π_2	0	1	0	0	0	0	0	1	2	0
π_3	0	0	1	0	0	0	0	1	1	1
π_4	0	0	0	1	0	0	0	1	1	0
π_5	0	0	0	0	1	0	0	1	1	0
π_6	0	0	0	0	0	1	0	1	1	0
π_7	0	0	0	0	0	0	1	1	1	-1

- A & B just filled in per dimensions of parameters
- D is always identity

Completing the Dimensional Matrix

- We now need to complete the definition of the dimensionless variables
- C Block was determined as

$$-\left(A^{-1}B\right)^{T} = -\left[\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}\right]^{T}$$

Reading Out Dimensionless Variables

	λ	Π_h	Π_v	μ_h	μ_v	α	β_h	β_v	S_{h0}	S_{v0}
Host	0	1	0	0	0	0	0	-1	1	0
Vector	0	0	1	0	0	0	-1	0	0	1
Time	-1	-1	-1	-1	-1	-1	-1	-1	0	0
π_1	1	0	0	0	0	0	0	1	1	0
π_2	0	1	0	0	0	0	0	1	2	0
π_3	0	0	1	0	0	0	0	1	1	1
π_4	0	0	0	1	0	0	0	1	1	0
π_5	0	0	0	0	1	0	0	1	1	0
π_6	0	0	0	0	0	1	0	1	1	0
π_7	0	0	0	0	0	0	1	1	1	-1

$$\pi_1 = \lambda \beta_v S_{h_0}$$

$$\pi_2 = \Pi_h \beta_v \left(S_{h_0} \right)^2$$

$$\pi_3 = \Pi_v \beta_v S_{h_0} S_{v_0}$$

$$\pi_4 = \mu_h \beta_v S_{h_0}$$

$$\pi_5 = \mu_v \beta_v S_{h_0}$$

$$\pi_6 = \alpha \beta_v S_{h_0}$$

$$\pi_7 = \beta_h \beta_v \frac{S_{h_0}}{S_{v_0}}$$

Identifying the Model Laws

- Dimensionless variables must hold identical values for full-sized and reduced-size models

$$\lambda^* \beta_v^* S_{h_0}{}^* = \lambda \beta_v S_{h_0}$$

$$\Pi^*{}_h \beta^*{}_v \left(S^*{}_{h_0}\right)^2 = \Pi_h \beta_v \left(S_{h_0}\right)^2$$

$$\Pi^*{}_v \beta^*{}_v S^*{}_{h_0} S^*{}_{v_0} = \Pi_v \beta_v S_{h_0} S_{v_0}$$

$$\mu^*{}_h \beta^*{}_v S^*{}_{h_0} = \mu_h \beta_v S_{h_0}$$

$$\mu^*{}_v \beta^*{}_v S^*{}_{h_0} = \mu_v \beta_v S_{h_0}$$

$$\alpha \beta^*{}_v S^*{}_{h_0} = \alpha \beta_v S_{h_0}$$

$$\beta^*{}_h \beta^*{}_v \frac{S^*{}_{h_0}}{S^*{}_{v_0}} = \beta_h \beta_v \frac{S_{h_0}}{S_{v_0}}$$

Solving the Scaling Relations

- Assume that all parameters of the reduced-size model are some coefficient times the corresponding parameters of the full-size model e.g. $\beta_v^* = c\beta_v$

$$S^*_{h_0} = cS_{h_0}$$

- Cancelling the terms for the full-size model vars, we get products (quotients) of the coefficients e.g.

$$\lambda^* \beta_v^* S_{h_0}^* = \lambda \beta_v S_{h_0} \quad \Rightarrow \quad c_\lambda c_{\beta_v} c_{S_{h_0}} = 1$$

$$\Pi^*_h \beta^*_v \left(S^*_{h_0} \right)^2 = \Pi_h \beta_v \left(S_{h_0} \right)^2 \quad \Rightarrow \quad c_{\Pi_h} c_{\beta_v} \left(c_{S_{h_0}} \right)^2 = 1$$

- Taking the logarithm of both sides

$$c_\lambda c_{\beta_v} c_{S_{h_0}} = 1 \quad \Rightarrow \quad \ln c_\lambda + \ln c_{\beta_v} + \ln c_{S_{h_0}} = 0$$

$$c_{\Pi_h} c_{\beta_v} \left(c_{S_{h_0}} \right)^2 = 1 \quad \Rightarrow \quad \ln c_{\Pi_h} + \ln c_{\beta_v} + 2 \ln c_{S_{h_0}} = 0$$

Completing the Experiment

- We choose 2 parameters for the scale model
  - We pick S_{h0} and S_{v0} (Host and Vector Populations)
  - These are the free variables (gaussian elimination)
  - All remaining parameters of the reduced-size model can be determined from these variables
- We can now run the experiment using the reduced-size model & make measurements of λ_r.
- Finally, we can compute the value of λ_f from the measured value for λ_r.

Power Law Scaling and the Buckingham Pi theorem

$$\Pi_1 = f\left(\Pi_2 \Pi_3 \cdots \Pi_n\right)$$

Recall that Π_1 is dimensionless parameter 1:

$$\Pi_1 \equiv a_1^{i_1} a_2^{i_2} a_3^{i_3} \cdots a_d^{i_d}$$

Where a are the original parameters

And thus

$$a_1 = a_2^{\frac{i_2}{i_1}} a_3^{\frac{i_3}{i_1}} \cdots a_d^{\frac{i_d}{i_1}} f\left(\Pi_2 \Pi_3 \cdots \Pi_n\right)$$

Power Law Scaling & Log-Log Graphs

- $y=x^a$
- $\log y = a \log x$
- If x is negative, have something like